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The f-index of inclusion as optimal adjoint pair for fuzzy modus ponens

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Abstract

We continue studying the properties of the f-index of inclusion and show that, given a fixed pair of fuzzy sets, their f-index of inclusion can be linked to a fuzzy conjunction which is part of an adjoint pair. We also show that, when this pair is used as the underlying structure to provide a fuzzy interpretation of the *modus ponens* inference rule, it provides the maximum possible truth-value in the conclusion among all those values obtained by fuzzy *modus ponens* using any other possible adjoint pair.

Keywords: Fuzzy logic; f-index of inclusion; Fuzzy inclusion; Inclusion measures; Residuated lattice; Galois connection

1. Introduction

This paper focuses on the notion of f-index, originally introduced to study the fuzzy inclusion between fuzzy sets, and its multiple applications within the theory of mathematical fuzzy logic (or fuzzy logic in the strict sense [17]). Most of the generalizations in the literature about fuzzy inclusion have a common feature; namely, they are fuzzy relations which, given a pair of fuzzy sets, A and B, assign a value in the unit interval that determines the degree of inclusion of A in B. In [29], we followed a different approach and introduced the notion of inclusion by assigning a *mapping* to each pair of fuzzy sets.

The origin of the f-index of inclusion can be dated back to the incorporation of negation connectives [38] in multi-adjoint logic programs [30,31]. With the incorporation of negations in logic programs arises an important feature: inconsistency. In order to handle the inconsistency of fuzzy logic programs [22] within our research line on normal residuated (i.e. mono-adjoint) logic programs under the answer-set semantics [23], we introduced the notions of *coherence* [11]. As a result, it was clear that consistency (or inconsistency) should not be considered as a crisp notion when applied in (general) fuzzy logic theories, and different approaches were proposed for this goal [25]. As a result, we introduced the notion of *weak-contradiction* in [5] as a generalization of the notion of coherence in the general framework of fuzzy set theory.

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Encouraged to search some motivational aspects to the notion of *weak-contradiction*, we realised that it was closely related to a kind of inclusion. Hence, soon after introducing measures for weak-contradiction, we started to imagine some kind of *function-based* approach to measuring the inclusion between fuzzy sets, and presented the first ideas about the f-index of inclusion in [29]. Later, in [24], we analyzed the f-index of inclusion in terms of the four most common axiomatic definitions of measure of inclusion namely, Sinha-Dougherty [36], Kitainik [19], and Fan-Xie-Pei [13]. We showed that it satisfies the well-known axioms proposed in these references, except the relationship with the complementary namely, the degree of inclusion of one fuzzy set A into another B should coincide with the degree of inclusion of B^c into A^c. In this respect two solutions were proposed: in [24] this was solved by rewriting the axiom in terms of Galois connections [8], and in [27] by means of a *measure* of inclusion defined in terms of the f-index of inclusion. Once the f-index of inclusion had been consolidated, we have recently recovered the idea of relating the two research lines emerged in parallel, namely the weak-contradiction and the f-index of inclusion, with satisfactory results [26,28].

In this paper we continue analyzing the relationship of the f-index of inclusion with the standard fuzzy measures of inclusion based on residuated implication [14]. Section 2 contains the necessary definitions for introducing the f-index of inclusion. In Section 3 we show that, on the one hand, the f-index of inclusion corresponding to a fixed pair of fuzzy sets can be linked to a standard fuzzy inclusion measure defined by means of an adjoint pair (different fuzzy sets, different adjoint pairs). Taking into account that an adjoint pair is the key notion within residuated lattices, as they allow to reproduce the *modus ponens* inference rule in a fuzzy logic [17], we conclude that the f-index of inclusion is related to the fuzzy *modus ponens*. Conversely, we have that every standard fuzzy inclusion measure defined by means of an adjoint pair can be considered a particular f-index of inclusion. As a result, we can state that the f-index of inclusion somehow groups together all the adjoint pairs and chooses one to define a standard measure of inclusion. Actually, we show in Section 4 that this adjoint pair is optimal with respect to a fuzzy *modus ponens*. Accordingly, the f-index of inclusion seems to be suitable in approximate reasoning, as it has been already done with the standard fuzzy measures of inclusion [10]. Finally, in Section 5 we present some conclusions and future works.

2. Preliminary definitions

2.1. Mathematical fuzzy logic

Fuzzy logic is based on degrees of truth. Different algebras of truth-values generate different logics, with different axiomatisations and even different inference rules. A general approach to fuzzy logic allows to use different operators that extend the classical logical connectives; in this setting, it is worth mentioning that the conjunction operator is usually represented by a *t-norm* $*$ [9], namely, a binary mapping $*\colon [0,1]\times[0,1]\to[0,1]$ which is commutative, associative, and monotone, for which 1 is the unit element. Concerning fuzzy implications, they are usually connected to the fuzzy conjunction in terms of residuation. Let us recall that for any left continuous t-norm $*$ there is a unique $\to_*\colon [0,1]\times[0,1]\to[0,1]$ such that for all $x,y,z\in[0,1]$ the following equivalence holds

$$z*x\le y \iff z\le x\to_* y \tag{1}$$

Note, however, that there are situations in which conjunctions need not be, for instance, commutative [16]. In order to obtain a flexible framework which accommodates most of these possibilities, we introduce below the most general definition of fuzzy conjunction and fuzzy implication.

Definition 1. An operator $*\colon [0,1]\times[0,1]\to[0,1]$ is called fuzzy conjunction if $*$ is monotonic in each component and satisfies the boundary conditions $0*0=0*1=1*0=0$ and $1*1=1$.

An operator $\to\colon [0,1]\times[0,1]\to[0,1]$ is called fuzzy implication if $\to$ is antitonic in the first component, monotonic in the second component and satisfies the boundary conditions $0\to 0=0\to 1=1\to 1=1$ and $1\to 0=0$.

It is worth mentioning that residuation can also be defined on a fuzzy conjunction in the sense of the previous definition. Property (1) is often called *adjointness property*, and the pair $(*,\to_*)$ is called *adjoint pair*. The notion of residuum is closely related to the notion of isotone Galois connection [7] (also called adjunction [6]), that is used in this paper.

Definition 2. Let A and B be lattices, a pair of mappings $f: A \to B$ and $g: B \to A$ is said to be an *(isotone) Galois connection* if the following equivalence holds for all $a \in A, b \in B$

$$f(a) \leq b \iff a \leq g(b)$$

2.2. Defining the f-index of inclusion on fuzzy sets

A fuzzy set A is a pair $(\mathscr{U}, \mu_A)$ where $\mathscr{U}$ is a non-empty set (called universe of A) and μ_A is a mapping from $\mathscr{U}$ to $[0, 1]$ (called membership function of A). In general, the universe is a fixed set for all the fuzzy sets considered and, therefore, each fuzzy set is determined by its membership function. For the sake of clarity, we identify fuzzy sets with their membership functions (i.e., $A(u) = \mu_A(u)$) and omit the universe if not necessary. On the set of fuzzy sets defined on the universe $\mathscr{U}$, denoted $\mathscr{F}(\mathscr{U})$, we can extend the usual crisp operations of union, intersection and complement as follows: given two fuzzy sets A and B, we define

- (union) $(A \cup B)(u) = \max\{A(u), B(u)\}$
- (intersection) $(A \cap B)(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = 1 - A(u)$.

The previous extensions of union, intersection and complement are the most common in the literature but, certainly, there are other options. For example, as generalization of the previous extensions, many authors use t-norms to generalize intersection, t-conorms to generalize union and negation operators to generalize the complement.

The *Zadeh's inclusion* [40] of a fuzzy set A into another fuzzy set B is defined as $A(u) \leq B(u)$ for all $u \in \mathscr{U}$. This is a crisp notion that was later extended to the so-called *fuzzy inclusion degree* of a fuzzy set A into a fuzzy set B, denoted by $S(A, B)$, as follows [2]:

$$S(A, B) = \bigwedge_{u \in \mathscr{U}} A(u) \to B(u). \tag{2}$$

The f-index of inclusion quantifies the inclusion of one fuzzy set into another by a mapping from $[0, 1]$ to $[0, 1]$. Actually, this feature is an important difference from the standard approach given in Equation (2) and other definitions or procedures to represent the inclusion in a fuzzy environment [36,19,13,39], where the inclusion of one fuzzy set into another is given, in general, by a value in the unit interval $[0, 1]$. Not any mapping from $[0, 1]$ to $[0, 1]$ can be used to represent the f-index of inclusion: the set of possible assignable mappings is introduced below, together with the basic notion of f-inclusion.

Definition 3 *([29]).*

- The set of *indexes of inclusion*, denoted by Ω, consists of all the deflationary and monotonically increasing mapping; that is, any mapping $f: [0, 1] \to [0, 1]$ such that
  - $f(x) \leq x$ for all $x \in [0, 1]$ and
  - $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in [0, 1]$.
- Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that A *is f-included in* B (denoted by $A \subseteq_f B$) if and only if the inequality $f(A(u)) \leq B(u)$ holds for all $u \in \mathscr{U}$.

Note that, fixed $f \in \Omega$, the relation of f-inclusion is a crisp relation and, in general, is not even an ordering relation (since transitivity fails). Therefore, for the same reason that Zadeh's inclusion, the f-inclusion (with a fixed $f \in \Omega$) seems to be unsuitable to represent the inclusion between two fuzzy sets; i.e. because of its lack of graduality. However, we do not define the f-index of inclusion by fixing one f-inclusion, but we consider all of them as different degrees of inclusion. Note that, since each f-inclusion is determined by a mapping f in Ω, we can identify the set of all the possible f-inclusion relations with Ω. The consideration of Ω as an appropriate set of indexes of inclusion is described and motivated in detail in [29] and can be summarized in the following items:

- Ω has the structure of complete lattice with the natural ordering between functions; i.e., given $f, g \in \Omega$, we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$. In particular, the mappings id (defined by $id(x) = x$ for all $x \in [0,1]$) and $\bot$ (defined by $\bot(x) = 0$ for all $x \in [0,1]$) are the top and bottom elements in Ω, respectively.
- Each $f \in \Omega$ determines a restriction, via the corresponding f-inclusion, that can be understood as "*how much we have to reduce the truth-values of a fuzzy set in order to be included into another in Zadeh's sense*". Thus, each $f \in \Omega$ can be seen as a measure of how much Zadeh's inclusion is violated.
- Finally, the greater the mapping $f \in \Omega$ the stronger the restriction imposed by the f-inclusion. In particular, the id-inclusion is the most restrictive case (and is equivalent to Zadeh's inclusion) and the $\bot$-inclusion does not establish any restriction at all.

The f-index of inclusion is based on the idea "the more f-inclusions holding between two sets, the greater is the fuzzy inclusion". Fortunately, we do not need to check all the f-inclusions between two sets because, given two fuzzy sets, the subset $\Lambda(A, B) = \{f \in \Omega \mid A \subseteq_f B\}$ has a maximum element in Ω. This fact allows us to introduce the following definition.

Definition 4 *(f-index of inclusion [29]).* Let A and B be two fuzzy sets, the *f-index of inclusion* of A in B, denoted by $Inc(A, B)$, is defined as the maximum of $\Lambda(A, B)$, that is $Inc(A, B) = \max\{f \in \Omega \mid A \subseteq_f B\}$.

Note firstly that the f-index of inclusion of A in B does not depend on any prior assumption or any kind of parameter [14]. Secondly, thanks to the properties of the f-inclusion (see [29]), considering the whole set of mappings $f \in \Omega$ such A is f-included in B (i.e., $\Lambda(A, B)$) and the f-index of inclusion (i.e., $Inc(A, B)$), are equivalent, since $\Lambda(A, B)$ is characterized by $Inc(A, B)$. Lastly, in [28], an analytical expression for $Inc(A, B)$ together with some properties that motivate the use of $Inc(A, B)$ as a suitable index of inclusion between two fuzzy sets can be found. These results are recalled in the form of the two theorems below:

Theorem 1 *([28]). Let A and B be two fuzzy sets, then* $Inc(A, B)(x) = \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\} \wedge id$.

Theorem 2 *([28]). Let A, B and C be fuzzy sets,*

1. *(Full inclusion)* $Inc(A, B) = id$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.
2. *(Null inclusion)* $Inc(A, B) = \bot$ *if and only if there is a set of elements in the universe* $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ *such that* $A(u_i) = 1$ *for all* $i \in I$ *and* $\bigwedge_{i \in I} B(u_i) = 0$.
3. *(Pseudo transitivity)* $Inc(B, C) \circ Inc(A, B) \leq Inc(A, C)$.
4. *(Monotonicity) If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then,* $Inc(C, A) \leq Inc(B, A)$.
5. *(Monotonicity) If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then,* $Inc(A, B) \leq Inc(A, C)$.
6. *(Transformation Invariance) Let* $T : \mathcal{U} \to \mathcal{U}$ *be a bijective mapping*[1] *on* $\mathcal{U}$*, then* $Inc(A, B) = Inc(T(A), T(B))$.
7. *(Relationship with intersection)* $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.
8. *(Relationship with union)* $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.

The previous properties were originally studied in [24] aimed at comparing the f-index of inclusion with the main axiomatic approaches in the literature concerning fuzzy measures of inclusions. Actually, we can state that the f-index of inclusion is almost a Fan-Xie-Pei [13], a Sinha-Dougherty [36] and a Kitainik [19] measure of inclusion. The only property encouraged by the previous authors that is not included in Theorem 2 is the one related to the complement and contraposition rule. Such a property states that the degree of inclusion between a fuzzy set A into another B should coincide with the degree of inclusion between B^c and A^c. In general, $Inc(A, B) \neq Inc(B^c, A^c)$, but in [27] it was shown that such a property is also captured by the f-index of inclusion by means of a *measure* of inclusion; i.e., by a mapping that assigns to each f-index of inclusion a value in the unit interval $[0, 1]$. Finally, we recall the following proposition, that establishes a relationship between the f-inclusion and the complement of fuzzy sets via an adjointness property, that will be used later in Section 3.

[1] Usually, in approaches related to measures of inclusion, e.g., [28,36], bijective mappings in $\mathcal{U}$ are called transformations.

Proposition 1 *([24]). Let (f, g) be an isotone Galois connection, then $A \subseteq_f B$ if and only if $B^c \subseteq_{1-g(1-x)} A^c$.*

3. The f-index of inclusion in terms of residuated implications

The aim of this section is to relate the f-index of inclusion to residuated implications. It is easy to check that the standard fuzzy inclusion in Equation (2) where $\to$ is a residuated implication, is equivalent to a particular degree of f-inclusion. Specifically, given an adjoint pair $(*, \to)$, we have for all $\alpha \in [0, 1]$ the equivalence

$$A(u) \to B(u) \geq \alpha \iff A(u) * \alpha \leq B(u).$$

Therefore, if we consider the mapping $f_\alpha(x) = x * \alpha$, which obviously belongs to Ω, we have:

$$\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \geq \alpha \iff A(u) \to B(u) \geq \alpha \text{ for all } u \in \mathcal{U} \iff A \text{ is } f_\alpha\text{-included in } B$$

As a result, the inequality $S(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \geq \alpha$ is equivalent to say that A is f_α-included in B. Therefore, the standard procedure to assign truth degrees to a formula of the type $\forall u \big(A(u) \to B(u)\big)$ by means of adjoint pairs, coincides with a specific case of f-inclusion. The previous results lead us to make a slight modification in the definition of the f-index of inclusion in order to obtain an actual generalization of the degrees of inclusion based on a pre-fixed adjoint pair. Specifically, we can define a variation of the f-index of inclusion restricted to a certain set of indexes of inclusion as follows:

Definition 5. Let A and B be two fuzzy sets and Θ be a join-sublattice of Ω; i.e., Θ is closed under arbitrary suprema and contains $\bot$ and id. Then, the f-index of inclusion restricted to Θ, denoted by $Inc_\Theta(A, B)$, is defined as

$$Inc_\Theta(A, B) = \sup\{f \in \Theta \mid A \subseteq_f B\}.$$

Note that $Inc_\Theta(A, B)$ is well defined, since we require Θ to be closed under arbitrary suprema; i.e., $\bigvee_{i \in \mathbb{I}} f_i \in \Theta$ for all subset $\{f_i\}_{i \in \mathbb{I}} \subseteq \Theta$. It is easy to check that the properties 3, 4, 5 and 6 of Inc given in Theorem 2 are satisfied also by Inc_Θ; the rest can be rewritten accordingly. We display below some examples of constructions of subsets of f-indexes of inclusion Θ according to the three most well-known t-norms.

Example 1. Let us consider the Gödel t-norm given by $x *_G y = x \wedge y$. Then, we define the Gödel set of f-index of inclusions Θ_G as the set $\Theta_G = \{x *_G \alpha \mid \alpha \in [0, 1]\}$. Note that for each mapping $f_\alpha^G \in \Theta_G$ there exists $\alpha \in [0, 1]$ such that

$$f_\alpha^G(x) = \begin{cases} x & \text{if } x \leq \alpha \\ \alpha & \text{otherwise,} \end{cases}$$

and that given $\alpha, \beta \in [0, 1]$ we have that $f_\alpha^G \wedge f_\beta^G = f_{\alpha\wedge\beta}^G$ and $f_\alpha^G \vee f_\beta^G = f_{\alpha\vee\beta}^G$. As a result, Θ_G inherits the lattice structure of $[0, 1]$ and therefore Θ_G is complete and totally ordered.

A similar construction can be done by considering product and Łukasiewicz t-norms instead of the Gödel one. Accordingly, we have the set of f-index of inclusions Θ_P and Θ_L given by $\Theta_P = \{x *_P \alpha \mid \alpha \in [0, 1]\}$ and $\Theta_L = \{x *_L \alpha \mid \alpha \in [0, 1]\}$. Curiously enough, Θ_P and Θ_L inherit the lattice structure of $[0, 1]$ in the same way than Θ_G. In other words, given $f_\alpha^L \in \Theta_L$ (resp. $f_\alpha^P \in \Theta_P$) there exists $\alpha \in [0, 1]$ such that

$$f_\alpha^L(x) = \begin{cases} 0 & \text{if } x \leq \alpha - 1 \\ x + \alpha - 1 & \text{otherwise.} \end{cases}$$

(resp. $f_\alpha^P = x \cdot \alpha$) and given $\alpha, \beta \in [0, 1]$ we have that $f_\alpha^L \wedge f_\beta^L = f_{\alpha\wedge\beta}^L$ and $f_\alpha^L \vee f_\beta^L = f_{\alpha\vee\beta}^L$ (resp. $f_\alpha^P \wedge f_\beta^P = f_{\alpha\wedge\beta}^P$ and $f_\alpha^P \vee f_\beta^P = f_{\alpha\vee\beta}^P$). As a conclusion, Θ_L and Θ_P inherit the lattice structure of $[0, 1]$ and therefore are complete and totally ordered.

The three sets of indexes of inclusion built in Example 1 just consider combinations of linear mappings. The following example presents a simple non-linear set of f-indexes of inclusion on which Definition 5 can be applied.

Example 2. Let us consider the set of f-indexes of inclusion $\Theta = \{x^n \mid n \in \mathbb{N}\} \cup \{\bot\}$ where x^n denotes the standard power of n. Note that we can identify each element in Θ with an element in $\mathbb{N} \cup \{\infty\}$; where $\bot$ is identified with x^∞. Moreover, Θ inherits the complete lattice structure of $\mathbb{N} \cup \{\infty\}$ with respect to the dual ordering; specifically Θ is a complete sublattice of Ω and actually, $x^n \wedge x^m = x^{n \vee m}$ and $x^n \vee x^m = x^{n \wedge m}$ for $n, m \in \mathbb{N} \cup \{\infty\}$.

The following result shows that $Inc_\Theta(A, B)$ effectively generalizes the standard measure of inclusion given by Equation (2) when it is based on adjoint pairs.

Theorem 3. *Let* $(*, \to)$ *be an adjoint pair and* $\Theta = \{x * \alpha \mid \alpha \in [0, 1]\}$ *then, for all pairs of fuzzy sets A and B,*

$$Inc_\Theta(A, B)(x) = x * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right).$$

Proof. It is not difficult to check that Θ is a complete lattice: to begin with, the bottom and top elements $\bot, id$ belong to Θ. Moreover, since $[0, 1]$ is a complete lattice and Θ inherits the lattice structure of $[0, 1]$, the lattice Θ is complete as well.

Given a pair of fuzzy sets A and B, since $Inc_\Theta(A, B) \in \Theta$, there exists at least one $\alpha \in [0, 1]$ such that $Inc_\Theta(A, B)(x) = x * \alpha$. We choose δ as the supremum of all those α's, that is, $\delta = \sup\{\alpha \in [0, 1] \mid Inc_\Theta(A, B)(x) = x * \alpha\}$. Let us prove now that $Inc_\Theta(A, B)(x) = x * \delta$, i.e., that such a supremum is actually a maximum. Let $x \in [0, 1]$, then:

$$x * \delta = x * \sup\{\alpha \mid Inc_\Theta(A, B)(x) = x * \alpha\} = \sup\{x * \alpha \mid Inc_\Theta(A, B)(x) = x * \alpha\} = Inc_\Theta(A, B)(x)$$

To finish the proof we only have to show that $\delta = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)$. Since A is $Inc_\Theta(A, B)$-included in B, for all $u \in \mathcal{U}$ we have:

$$Inc_\Theta(A, B)(A(u)) \leq B(u) \iff A(u) * \delta \leq B(u) \iff \delta \leq A(u) \to B(u)$$

and then $\delta \leq \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)$. On the other hand, to prove that $\delta \geq \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)$ we proceed by reductio ad absurdum and assume that $\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) = \gamma > \delta$. Then, since $(*, \to)$ is an adjoint pair, we have that:

$$\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) = \gamma \quad \Longrightarrow \quad A(u) \to B(u) \geq \gamma \ \text{ for all } u \in \mathcal{U} \quad \iff \quad A(u) * \gamma \leq B(u) \ \text{ for all } u \in \mathcal{U}.$$

As a result, A is $(x * \gamma)$-included in B, which contradicts that $Inc_\Theta(A, B)(x) = x * \delta$ and the choice of δ. □

The following straightforward corollary shows that the f-index of inclusion defined under the parameters of the previous theorem, can be used to retrieve the value of $S(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)$.

Corollary 1. $(*, \to)$ *be an adjoint pair such that* $1 * x = x$ *for all* $x \in [0, 1]$ *and* $\Theta = \{x * \alpha \mid \alpha \in [0, 1]\}$ *then, for all pair of fuzzy sets A and B,*

$$Inc_\Theta(A, B)(1) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)$$

From the previous results, we can say that the original f-index of inclusion takes into consideration, at least, the whole set of possible adjoint pairs and their respective degrees of inclusion given in Equation (2).

In the following, we show a kind of converse statement namely, when the universe is finite, the original f-index of inclusion can be always linked to an adjoint pair. In order to prove such a relationship, we need to introduce firstly the following theorem.

Theorem 4. *Let A and B be two fuzzy sets on a finite universe* $\mathcal{U}$. *Then, there exists a mapping* $\overline{Inc}(A, B)\colon [0, 1] \to [0, 1]$ *such that* $(Inc(A, B), \overline{Inc}(A, B))$ *forms an adjoint pair.*

Proof. We will use the following characterization of existence of adjoint pair: given a monotonic mapping $f: [0,1] \to [0,1]$, there exists a mapping $g: [0,1] \to [0,1]$ such that (f, g) forms an adjoint pair if and only if $f(0) = 0$ and $f\left(\bigvee X\right) = \sup_{x\in X}\{f(x)\}$ for all $X \subseteq [0,1]$ (see [15, Proposition 9]).

Firstly, note that $Inc(A, B)$ is monotonic and it holds $Inc(A, B)(0) = 0$, since it belongs to Ω. Then, we only have to prove that $\sup_{x\in X}\{Inc(A, B)(x)\} = Inc(A, B)\left(\bigvee X\right)$ for all $x \in X$.

Let $X \subseteq [0,1]$. By the monotonicity of $Inc(A, B)$, we have that $Inc(A, B)(x) \le Inc(A, B)\left(\bigvee X\right)$ for all $x \in X$ and then $\sup_{x\in X}\{Inc(A, B)(x)\} \le Inc(A, B)\left(\bigvee X\right)$. In order to prove the other inequality, note that if $\bigvee X \in X$, then the proof is straightforward. So, let us assume that $\bigvee X \notin X$. It is enough to prove that there exists $x_0 \in X$ such that

$$\{B(u) \mid x_0 \le A(u)\ ,\ u \in \mathcal{U}\} = \left\{B(u) \mid \bigvee X \le A(u)\ ,\ u \in \mathcal{U}\right\}$$

since, as a result, by means of Theorem 1, we would have

$$\begin{aligned}\sup_{x\in X}\{Inc(A, B)(x)\} &\ge Inc(A, B)(x_0) = \sup\{B(u) \mid x_0 \le A(u)\}\\ &= \sup\left\{B(u) \mid \bigvee X \le A(u)\right\} = Inc(A, B)\left(\bigvee X\right),\end{aligned}$$

and the proof would end.

Consider an increasing chain $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_n x_n = \bigvee X$, as a result we have the following corresponding chain of sets

$$\{B(u) \mid x_1 \le A(u)\} \supseteq \{B(u) \mid x_2 \le A(u)\} \supseteq \cdots \supseteq \{B(u) \mid x_n \le A(u)\} \supseteq \ldots$$

Since $\mathcal{U}$ is finite, the sequence of sets should be eventually constant for all n greater than certain integer k. Let us prove that $\{B(u) \mid x_k \le A(u)\} = \left\{B(u) \mid \bigvee X \le A(u)\right\}$.

By reductio ad absurdum, let us assume that there exists $u_0 \in \mathcal{U}$ such that $B(u_0) \in \{B(u) \mid x_k \le A(u)\}$ but $B(u_0) \notin \left\{B(u) \mid \bigvee X \le A(u)\right\}$. This means that $x_k < A(u_0) < \bigvee X$ and, as a result, there exists x_j in the chain such that $x_k < A(u_0) < x_j < \bigvee X$, but then we would have $\{B(u) \mid x_k \le A(u)\} \supset \{B(u) \mid x_j \le A(u)\}$ which contradicts the choice of k. □

As an important consequence of the previous result, in the case of a finite universe, we can restrict the set of indexes and still, retrieve the same original f-index of inclusion.

Corollary 2. *Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$ and let $\mathcal{G}$ be the set of all the mappings in $f \in \Omega$ such that there exists $g: [0,1] \to [0,1]$ such that (f, g) forms an adjoint pair. Then*

$$Inc(A, B) = Inc_{\mathcal{G}}(A, B)$$

Proof. This result is a consequence of Theorem 4. □

The following theorem has important repercussions, since it shows that in the case of a finite universe, the f-index of inclusion can be associated to at least one adjoint pair formed by a fuzzy conjunction and a fuzzy implication.

Theorem 5. *Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$. Then, there exists an adjoint pair $(*, \to)$ with $*$ a commutative fuzzy conjunction and $\alpha \in [0,1]$ such that*

$$Inc(A, B)(x) = x * \alpha.$$

Proof. Let $\alpha = Inc(A, B)(1)$ and let us prove that for the fuzzy conjunction $*$ defined by

$$x * y = \begin{cases} 0 & \text{if } \alpha < x \vee y \text{ and } x \wedge y = 0\\ 1 & \text{if } \alpha < x \wedge y\\ Inc(A, B)(x \wedge y) & \text{if } x \vee y \le \alpha\\ Inc(A, B)(x \vee y) & \text{otherwise}\end{cases} \tag{3}$$

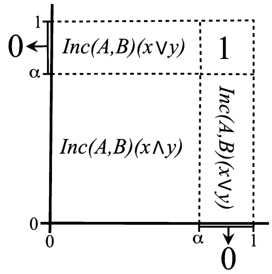

Fig. 1. Diagram with the description of the conjunction $*$ defined in the proof.

there exists an implication $\to$ such that $(*, \to)$ forms an adjoint pair. For the sake of presentation, we have included a diagram in Fig. 1, where the conjunction $*$ is depicted in the square $[0, 1]^2$.

Firstly, let us check that $x * y$ is a commutative conjunction. The commutativity is straightforward. For the monotonicity, let us assume $x, y_1, y_2 \in [0, 1]$ such that $y_1 \leq y_2$. If the pairs (x, y_1) and (x, y_2) belong to the same case in the definition $*$, then we have trivially $x * y_1 \leq x * y_2$. The rest of possibilities can be easily checked in Fig. 1. The only non-entirely trivial case is that of the segments in the X and Y axes. Assume $x \vee y_1 \leq \alpha$, and $\alpha < x \vee y_2$ and $x \wedge y_2 = 0$. We have two subcases: either $x = 0$ or $y_2 = 0$ (hence $y_1 = 0$), in both cases we have

$$x * y_1 = Inc(A, B)(x \wedge y_1) = Inc(A, B)(0) = 0 \leq x * y_2$$

Secondly, we prove that $*$ forms a part of an adjoint pair by checking left-continuity, i.e. given a fuzzy conjunction $*$ there exists an implication $\to$ such that $(*, \to)$ forms adjoint pair if and only if $\left(\bigvee X\right) * y = \bigvee_{x \in X}(x * y)$ for all set $X \subseteq [0, 1]$ (see [34]).

Consider $X \subseteq [0, 1]$ and $y \in [0, 1]$, by commutativity we can assume $\bigvee X \leq y$, and it is enough to consider the following four cases:

Case 1: $\bigvee X \leq y \leq \alpha$**.** By properties of supremum and infimum and the fact that $Inc(A, B)$ is sup-preserving (as checked in the proof of Theorem 4):

$$\bigvee X * y = Inc(A, B)\left(\bigvee X \wedge y\right) = Inc(A, B)\left(\bigvee_{x \in X}(x \wedge y)\right)$$
$$= \bigvee_{x \in X}\left(Inc(A, B)\,(x \wedge y)\right) = \bigvee_{x \in X}(x * y)$$

Case 2: $0 = \bigvee X \leq \alpha \leq y$**.** The result is straightforward: $\left(\bigvee X\right) * y = \bigvee_{x \in X}(x * y) = 0$.

Case 3: $\alpha < \bigvee X \leq y$**.** Then there exists $x' \in X$ such that $\alpha < x' \leq \bigvee X$. As a result we have:

$$\bigvee X * y = 1 = x' * y = \bigvee_{x \in X}(x * y)$$

Case 4: $\bigvee X \leq \alpha < y$**.** Then $x \leq \alpha < y$ for all $x \in X$ and as a result we have

$$\bigvee X * y = Inc(A, B)\,(y) = \bigvee_{x \in X} Inc(A, B)\,(x \vee y) = \bigvee_{x \in X}(x * y)$$

Finally, let us check the equality $Inc(A, B)(x) = x * \alpha$ for all $x \in [0, 1]$:

$$x * \alpha = \begin{cases} 0 & \text{if } \alpha < x \vee \alpha \text{ and } x \wedge \alpha = 0 \\ 1 & \text{if } \alpha < x \wedge \alpha \\ Inc(A, B)(x \wedge \alpha) & \text{if } x \vee y \leq \alpha \\ Inc(A, B)(x \vee \alpha) & \text{otherwise} \end{cases}$$
$$= \begin{cases} Inc(A, B)(x) & \text{if } x \leq \alpha \\ Inc(A, B)(x) & \text{if } \alpha < x \end{cases} = Inc(A, B)(x) \quad \square$$

The reader might wonder whether we can relate the f-index of inclusion with a t-norm, which is by far, the most prominent family of fuzzy conjunctions used in fuzzy logic. That remains as an open problem, since, as we show in the following example, the conjunction $*$ defined in Equation (3) need not be associative.

Example 3. Let us consider, on the universe $\mathcal{U} = \{u_1, u_2\}$, the following fuzzy sets A and B given by $A(u_1) = 0.4$, $A(u_2) = 1$, $B(u_1) = 0$ and $B(u_2) = 0.4$. Then, we have that

$$Inc(A, B)(x) = \begin{cases} 0 & \text{if } x \le 0.4 \\ 0.4 & \text{if } x > 0.4 \end{cases}$$

Since $Inc(A, B)(1) = 0.4$, the conjunction $*$ defined in the proof of Theorem 5 (i.e., in Equation (3)) is:

$$x * y = \begin{cases} 0 & \text{if } 0.4 < x \vee y \ \text{ and } \ x \wedge y = 0 \\ 1 & \text{if } 0.4 < x \wedge y \\ Inc(A, B)(x \wedge y) & \text{if } x \vee y \le 0.4 \\ Inc(A, B)(x \vee y) & \text{otherwise} \end{cases}$$

Then, on the one hand, we have $(0.4 * 0.5) * 0.5 = 0.4 * 0.5 = 0.4$ but, on the other hand, $0.4 * (0.5 * 0.5) = 0.4 * 0.4 = 0$. In other words, $*$ is not associative. □

The following result states that we can link an associative and commutative conjunction $*$ to $Inc(A, B)$ when $Inc(A, B)(1) = 1$.

Corollary 3. *Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$ such that $Inc(A, B)(1) = 1$. Then, there exists an adjoint pair $(*, \to)$ with $*$ an associative and commutative fuzzy conjunction, and $\alpha \in [0, 1]$ such that*

$$Inc(A, B)(x) = x * \alpha.$$

Proof. We consider the same commutative operator $*$ defined in Equation (3) (i.e., in the proof of Theorem 5). Since $\alpha = Inc(A, B)(1) = 1$, we have that $x * y = Inc(A, B)(x \wedge y)$, which is obviously associative. □

Even under the hypothesis of the previous corollary, we cannot guarantee that the resulting operation $*$ constructed in the proof of Theorem 5 is a t-norm. Specifically, 1 need not be a neutral element since we have that $x * 1 = Inc(A, B)(x)$ and then, $x * 1 = x$ if and only if $Inc(A, B)(x) = x$.

The most important consequence of Theorem 5 is that we can go from the f-index of inclusion to residuated implications and, then, to the standard modus ponens used in fuzzy logic.

Corollary 4. *Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. Then, there exists a residuated implication $\to : [0, 1]^2 \to [0, 1]$ such that, for all $u \in \mathcal{U}$,*

$$Inc(A, B)(\alpha) \le B(u) \quad \iff \quad \alpha \le A(u) \to B(u)$$

Proof. By Theorem 5 we know that there exists an adjoint pair $(*, \to)$ and $\alpha \in [0, 1]$ such that $Inc(A, B)(x) = x * \alpha$. Then, by the adjoint property:

$$Inc(A, B)(\alpha) \le B(u) \quad \iff \quad x * \alpha \le B(u) \quad \iff \quad A(u) \to B(u) \ge \alpha \quad \square$$

Corollary 5. *Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. Then, there exists an adjoint pair $(*, \to)$ such that*

$$Inc(A, B)(x) = x * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right)$$

Proof. Consequence of Theorems 3 and 5. □

4. The f-index of inclusion as an optimal operator for performing inferences by fuzzy modus ponens

It is worth mentioning one significant repercussion in fuzzy logic of the results presented in Section 3. Let us assume that we are in a context of machine learning where we have a data set of instances that relate different attributes [33]; without loss of generality, we can also assume that the (finite) set of instance is our universe $\mathcal{U}$ and that their attributes are fuzzy sets on $\mathcal{U}$. Given two attributes A and B, we wonder whether there is any relation of dependence among them in terms of *"the value of one implies certain value in the other"*. This relationship can be syntactically represented by means of the implication $A \Rightarrow B$ and modelled by the formula $\forall u\big(A(u) \to B(u)\big)$ in a certain fuzzy (first-order) logic. The semantics of such a formula depends on an adjoint pair $(*, \to)$, and its truth-degree is given by the standard measure of inclusion $S(A, B) = \bigwedge_{u\in\mathcal{U}} A(u) \to B(u)$ (see [35] for more details). Moreover, with the truth-degree of $\bigwedge_{u\in\mathcal{U}} A(u) \to B(u)$ and the truth-degree of an instance $A(u_0)$, for certain $u_0 \in \mathcal{U}$, we can apply modus ponens and infer information about the instance $B(u_0)$. If we identify the values of $\bigwedge_{u\in\mathcal{U}} A(u) \to B(u)$ and $A(u_0)$ with α and β (in [0, 1]), such an inference can be represented as follows:

$$\begin{array}{lll} & A \Rightarrow B & \equiv \alpha \\ & A(u) & \equiv \beta \\ \hline \therefore & B(u) & \geq \beta * \alpha \end{array} \tag{4}$$

and the soundness of such an inference is given by the adjoint property of the adjoint pair $(*, \to)$. That is, since

$$A(u_0) \to B(u_0) \geq \bigwedge_{u\in\mathcal{U}} A(u) \to B(u) = \alpha$$

then, by the adjoint property we have: $B(u_0) \geq A(u_0) * \left(\bigwedge_{u\in\mathcal{U}} A(u) \to B(u)\right) = \beta * \alpha$.

Let us rewrite now the modus ponens inference in terms of the f-index of inclusion. Firstly, we can model the implication $A \Rightarrow B$ by $Inc(A, B)$, and then, we can define the following modus ponens inference:

$$\begin{array}{lll} & A \Rightarrow B & \equiv Inc(A, B) \\ & A(u) & \equiv \beta \\ \hline \therefore & B(u) & \geq Inc(A, B)(\beta) \end{array} \tag{5}$$

The soundness of this latter inference is due to definition of the f-index of inclusion. That is, since A is $Inc(A, B)$-included in B, we have that $Inc(A, B)(A(u)) \leq B(u)$ for all $u \in \mathcal{U}$. In particular, if $A(u_0) = \beta$, we have that $Inc(A, B)(\beta) \leq B(u_0)$.

The point now is: what is the relationship between the two versions of modus ponens defined by (4) and (5)? The answer is that version (5) is the greatest inference that can be obtained from modus ponens for the instance $B(u_0)$ by considering, as variables, the whole set of adjoint pairs in version (4). The following theorem is used to support that assertion:

Theorem 6. *Let A and B two fuzzy sets defined on a finite universe $\mathcal{U}$ and let $u_0 \in \mathcal{U}$, then:*

$$B(u_0) \geq Inc(A, B)(A(u_0)) \geq A(u_0) * \left(\bigwedge_{u\in\mathcal{U}} A(u) \to B(u)\right)$$

for all pair of residuated pairs $(*, \to)$ and $x \in [0, 1]$.

Proof. Let us consider an arbitrary adjoint pair $(*, \to)$. Then, we have that the mapping

$$f_*(x) = x * \bigwedge_{u\in\mathcal{U}} A(u) \to B(u)$$

is increasing and $f_*(x) \leq x$; i.e., $f_* \in \Omega$. On the other hand, thanks to Corollary 5, we can say that there exists an adjoint pair $(\odot, \hookrightarrow)$ such that

$$Inc(A,B)(x) = x \odot \left(\bigwedge_{u \in \mathcal{U}} A(u) \hookrightarrow B(u) \right)$$

Now, by definition of the f-index of inclusion, necessarily $Inc(A,B) \geq f_*$; that is:

$$Inc(A,B)(x) = x \odot \left(\bigwedge_{u \in \mathcal{U}} A(u) \hookrightarrow B(u) \right) \geq x * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right)$$

Finally, given $u_0 \in \mathcal{U}$, and since $Inc(A,B)(A(u_0)) \leq B(u_0)$, we have that

$$B(u_0) \geq Inc(A,B)(A(u_0)) \geq A(u_0) * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right). \quad \square$$

On the basis of the previous theorem, we can reconsider the comparison between the two versions of modus ponens; i.e., Equations (4) and (5). Theorem 6 states that for any pair of fuzzy sets A, B, for any adjoint pair $(*, \to)$, and for all $u_0 \in \mathcal{U}$, we have the inequalities:

$$B(u_0) \geq Inc(A,B)(A(u_0)) \geq A(u_0) * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u) \right).$$

In other words, the result of the fuzzy modus ponens modelled by the f-index of inclusion, i.e. the value $Inc(A,B)(A(u_0))$ (MP version (5)) is greater than or equal to the result of the fuzzy modus ponens modelled by any adjoint pair $(*, \to)$, i.e., the value $A(u_0) * \left(\bigwedge_{u \in \mathcal{U}} A(u) \to B(u)\right)$ (MP version (4)) and it is closer to the real value of the instance $B(u_0)$. As a result, we can say that the f-index of inclusion chooses among all the adjoint pairs, the optimal one to perform the modus ponens inference.

Another interesting advantage that provides the use of the f-index of inclusion is the possibility of defining a modus tollens inference thanks to Theorem 4 and Proposition 1. From both results we can ensure that the f-index of inclusion between the complement of B and the complement of A is at least $1 - \overline{Inc}(A,B)(1-x)$, where $\overline{Inc}(A,B)$ is the only mapping in $[0,1] \to [0,1]$ such that $(Inc(A,B), \overline{Inc}(A,B))$ forms an adjoint pair. In other words, we can define the following modus tollens inference for all $u \in \mathcal{U}$:

$$\begin{array}{lll} & A \Rightarrow B & \equiv Inc(A,B) \\ & B^c(u) & \equiv \beta \\ \hline \therefore & A^c(u) & \geq 1 - \overline{Inc}(A,B)(1-\beta) \end{array} \tag{6}$$

As mentioned, the soundness of this latter inference lays on Theorem 4 and Proposition 1, since for all $u \in \mathcal{U}$ we can ensure that for all $u \in \mathcal{U}$ we have

$$A^c(u) \geq 1 - \overline{Inc}(A,B)(1 - B^c(u))$$

It is important to introduce two remarks in this context: firstly, note that the previous inequality is equivalent to

$$A(u) \leq \overline{Inc}(A,B)(B(u))$$

for all $u \in \mathcal{U}$ and, therefore, the modus tollens can be used to infer an upper bound for the value of A; secondly, the difference between performing the inference of $A^c(u)$ from the modus ponens plus $Inc(B^c, A^c)$ and from the modus tollens plus $Inc(A,B)$, is that in the latter we cannot ensure the use of the optimal residuated implication. Nevertheless, the reader can find out that the difference between the mappings $Inc(B^c, A^c)(x)$ and $1 - \overline{Inc}(A,B)(1-x)$ is a set of measure 0 (see [27]). Hence, the difference between the modus tollens and the inference performed by the modus ponens and $Inc(B^c, A^c)$, is negligible.

To finish this section, we include a example to illustrate a fuzzy inference system based on the f-index of inclusion and its comparison with an inference system based on fixed adjoint pairs.

Example 4. Let us consider the following dataset of 8 entries and two variables (named A and B):

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
A	0.3	0.1	0.8	0.5	0.6	1	0.2	0.5
B	0.7	0	0.8	0.3	0.7	1	0.4	0.6

Note that each entry has been identified with one element in $\mathscr{U} = \{u_1, u_2, \dots, u_8\}$ and that the variables A and B can be considered fuzzy sets defined on $\mathscr{U}$. Our goal is to establish a relationship between A and B in terms of inclusion. On the one hand, we begin compute the three standard degrees of inclusion with respect to the three most common connectives in fuzzy logic: Łukasiewicz, Gödel and Product:

$$S_Ł(A,B) = \bigwedge_{I=1}^{8} A(u_i) \to_Ł B(u_i) = 0.8 \qquad S_Ł(B,A) = \bigwedge_{I=1}^{8} A(u_i) \to_Ł B(u_i) = 0.6$$

$$S_G(A,B) = \bigwedge_{I=1}^{8} A(u_i) \to_G B(u_i) = 0 \qquad S_G(B,A) = \bigwedge_{I=1}^{8} A(u_i) \to_G B(u_i) = 0.2$$

$$S_P(A,B) = \bigwedge_{I=1}^{8} A(u_i) \to_P B(u_i) = 0 \qquad S_P(B,A) = \bigwedge_{I=1}^{8} A(u_i) \to_P B(u_i) = \frac{3}{7}$$

As stated above, those three degrees of inclusion can be identified with the truth-degree of the first order formulae $\forall u\big(A(u) \to B(u)\big)$ and $\forall u\big(B(u) \to A(u)\big)$, respectively. Thus, as mentioned above, they can be identified as the truth degrees of the fuzzy rules $A \Rightarrow B$ (resp. $B \Rightarrow A$) and be used to perform inference based on the standard fuzzy Modus Ponens.

On the other hand, the two respective f-indexes of inclusion can be obtained by using the expression given in Theorem 1. It is worth to stress here that the computation of the indexes of inclusion is almost straightforward.

$$Inc(A,B)(x) = \begin{cases} 0 & \text{if } x \le 0.1 \\ x & \text{if } 0.1 < x \le 0.3 \\ 0.3 & \text{if } 0.3 < x \le 0.5 \\ x & \text{if } 0.5 < x \le 1 \end{cases} \qquad Inc(B,A)(x) = \begin{cases} x & \text{if } x \le 0.2 \\ 0.2 & \text{if } 0.2 < x \le 0.4 \\ 0.3 & \text{if } 0.4 < x \le 0.7 \\ x & \text{if } 0.7 < x \le 1 \end{cases}$$

As explained above, we can also apply a modus ponens inference based on the f-indexes of inclusion. Hence, we also may say that we have two fuzzy rules $A \Rightarrow B$ and $B \Rightarrow A$.

At this point we want to analyze which pair of fuzzy rules is more precise in the representation of the relationship between A and B. To carry out such an analysis, we assume that some entries of B are missing and we calculate their truth degrees from those of A and the four pairs of fuzzy rules given above. Below we provide a comparison between the different results obtained after the application of each modus ponens. For the sake of a better understanding of the inference, we consider a new entry related to $u_9 \in \mathscr{U}$; so we assume that $A(u_9) = 0.7$ but $B(u_9)$ is unknown. The information we can infer about $B(u_9)$ from the premise $A(u_9) = 0.7$ and the four modus ponens are the following:

$$\begin{array}{lll} & A \Rightarrow_Ł B & \equiv 0.8 \\ & A(u_9) & \equiv 0.7 \\ \hline \therefore & B(u_9) & \ge 0.5 \end{array} \qquad \begin{array}{lll} & A \Rightarrow_G B & \equiv 0 \\ & A(u_9) & \equiv 0.7 \\ \hline \therefore & B(u_9) & \ge 0 \end{array} \qquad \begin{array}{lll} & A \Rightarrow_P B & \equiv 0 \\ & A(u_9) & \equiv 0.7 \\ \hline \therefore & B(u_9) & \ge 0 \end{array} \qquad \begin{array}{lll} & A \Rightarrow B & \equiv Inc(A,B) \\ & A(u_9) & \equiv 0.7 \\ \hline \therefore & B(u_9) & \ge 0.7 \end{array}$$

Note firstly, that the modus ponens based on Product and Gödel connectives does not provide any useful information, since $A \Rightarrow_G B = S_G(A,B) = 0$ and $A \Rightarrow_P B = S_P(A,B) = 0$. Secondly, note that the greatest lower bound for the value $B(u_9)$ is given by the modus ponens based on the f-index of inclusion. That is not by chance, it is because of Theorem 5; it holds for all $u \in \mathscr{U}$ and for all dataset considered at the beginning of the example.

Conversely, we may consider the fuzzy rule $B \Rightarrow A$ and repeat the same procedure above by assuming that one B is known but A unknown. However, that is pointless. It is definitely more valuable to use the modus tollens (Equation (6)) of the rule $B \Rightarrow A$ and still assume that B is unknown. The first step is to compute the only mapping from [0, 1] to [0, 1] such that $(Inc(B,A), \overline{Inc}(B,A))$ forms an adjoint pair. That is easy by using the formula $\overline{Inc}(B,A)(x) = \sup\{y \in [0,1] \mid Inc(A,B) \le x\}$ (a very well known property of Galois Connections [15]), which results on:

$$\overline{Inc}(B,A) = \begin{cases} x & \text{if } x < 0.2 \\ 0.4 & \text{if } 0.2 \le x < 0.3 \\ 0.7 & \text{if } 0.3 \le x < 0.7 \\ x & \text{if } 0.7 \le x \le 1 \end{cases}$$

Now, since $A(u_9) = 0.7$, we have that $A^c(u_9) = 1 - 0.7 = 0.3$ and we can apply the modus tollens inference of Equation (6):

$$\begin{array}{lll} & B \Rightarrow A & \equiv Inc(B, A) \\ & A^c(u_9) & \equiv 0.3 \\ \hline \therefore & B^c(u_9) & \ge 0.3 \end{array}$$

Now, we can use that $B^c(u_9) \ge 0.3$ is equivalent to $B(u_9) \le 0.7$ and, as a result, by joining this inequality with $B(u_9) \ge 0.7$ we can conclude that $B(u_9) \in [0.7, 0.7]$, that is, $B(u_9) = 0.7$.

Concerning the standard degrees of inclusion, only the connectives of Łukasiewicz admit a modus tollens with respect to the standard negation.[2] It is well known that $S_Ł(A^c, B^c) = S_Ł(B, A) = 0.6$. By applying modus tollens here, we obtain that $B^c(u_9) \ge A^c(u_9) *_Ł 0.6 = 0$ and then $B(u_9) \in [0.5, 1]$.

Below we present a table with all the inferences for all the entries in the training dataset (i.e., for those $u \in \{u_1, \dots, u_8\}$). Note that by Theorem 6, the inferences based on the f-index of inclusion are consistent and more precise that those obtained by means of any adjoint pair; in this case we only compare it with Łukasiewicz, Godel and product.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
A	0.3	0.1	0.8	0.5	0.6	1	0.2	0.5
approx. of B by f-index	[0.3, 0.7]	[0, 0.1]	[0.8, 0.8]	[0.3, 0.7]	[0.6, 0.7]	[1, 1]	[0.2, 0.4]	[0.3, 0.7]
approx. of B by Łuka.	[0.1, 0.7]	[0, 0.5]	[0.6, 1]	[0.3, 0.9]	[0.4, 1]	[0.8, 1]	[0, 0.6]	[0.3, 0.9]
approx. of B by Gödel	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
approx. of B by Prod.	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
B	0.7	0	0.8	0.3	0.7	1	0.4	0.6

Note that for u_3 and u_6 we obtain the most precise inference; specifically, $B(u_3) = 0.8$ and $B(u_6) = 1$. Actually, if we consider a new entry u such that $A(u) > 0.6$, then the inference based on the f-index of inclusion provides the result $B(u) = A(u)$. The reason is because in the training dataset we have that $A(u) = B(u)$ for all $u \in \{u_1, \dots, u_8\}$ with $A(u) > 0.6$.

5. Conclusions and future work

We have presented some new properties of the so-called f-index of inclusion. Specifically, we have shown some close relationships between the f-index of inclusion and the standard measures of inclusion based on adjoint pairs; e.g., those formed by t-norms $*$ and their residua. As a consequence of our results, we can assert that a modus ponens based on the f-index of inclusion performs the greatest inference among all those that can be performed by adjoint pairs and their standard measures of inclusion. As a result of this feature, we have obtained the following results:

- The modus ponens based on the f-index of inclusion is just as suitable to perform logical inferences as the standard one based on adjoint pairs.
- The f-index of inclusion is essentially based on the notion of f-inclusion, which simply consists in the ordering $f(A(u)) \le B(u)$ in [0, 1] for monotonic mappings f less or equal than the identity function. In fact, the f-index of inclusion is closely related to the multi-adjoint structures [31] but approaching the issue in a simpler way.
- The optimality of the chosen residuated implication supports the use of the f-index of inclusion in machine learning where the knowledge in databases is modelled by rules of fuzzy logic. Specifically, this fact encourages

[2] It is convenient to mention that product and Gödel connectives admit also a modus tollens, but that is by considering the drastic negation, which provides very limited information in general.

the use of the f-index of inclusion as the core for performing formal reasoning, as done by monotonic functions in [37], and to represent fuzzy ontologies, as done by standard fuzzy inclusions in [4] or by classical inclusion in [1].

As future work, Example 4 has presented a very preliminary version of a possible fuzzy inference system based on the f-index of inclusion and its use to recovering missing data. To complete such a system, it is necessary to check different possibilities in the construction technique and to test the results of data that has been not used in the training dataset. From a theoretical point of view, we will study the problem of constructing a t-norm $*$ compatible with the proof of Theorem 5. Furthermore, the fact that the f-index of inclusion behaves similarly to an implication opens the possibility to include it in a residuated structure. In such a case, we could use the f-index of inclusion directly as an implication and apply it in all those areas where the underlying structure is residuated, for instance, Fuzzy Formal Concept analysis [3,20,32], Fuzzy Mathematical Morphology [12,21], Fuzzy Logic [35], Fuzzy Logic Programming [18].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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