Applied Mathematics and Computation 443 (2023) 127784

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Implicit and implicit-explicit Lagrange-projection finite volume schemes exactly well-balanced for 1D shallow water system

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ARTICLE INFO

Article history:
Received 7 July 2022
Revised 9 December 2022
Accepted 13 December 2022
Available online 25 December 2022

Keywords:
Lagrangian-projection technique
IMEX schemes
Well-balanced
Shallow-water

ABSTRACT

In this paper we consider the Lagrange-Projection technique in the framework of finite volume schemes applied to the shallow water system. We shall consider two versions of the scheme for the Lagrangian step: one fully implicit and one implicit-explicit, based on how the geometric source term is treated. First and second order well-balanced versions of the schemes are presented, in which the water at rest solutions are preserved. This allows to obtain efficient numerical schemes in low Froude number regimes, as the usual CFL restriction driven by the acoustic waves is avoided.

1. Introduction

This paper is devoted to the design of first and second order exactly well-balanced implicit numerical schemes for the shallow water equations using the Lagrange-Projection approach. Here by exactly well-balanced scheme we mean a scheme that is able to exactly preserve the water at rest steady states.

We will consider the shallow water equations (SWE) given by

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x \left(hu^2 + g\frac{h^2}{2} \right) = -gh\partial_x z, \end{cases} \tag{1.1}$$

where $z(x)$ denotes a given smooth topography and $g > 0$ is the gravity constant. The primitive variables are the water depth $h \geq 0$ and the velocity u, both depending on the space and time variables, $x \in \mathbb{R}$ and $t \in [0, \infty)$. In what follows we shall refer to the variables (x, t) as Eulerian coordinates, opposed to the Lagrangian coordinates that will be introduced

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https://doi.org/10.1016/j.amc.2022.127784

afterwards. We will refer to (1.1) as SWE in Eulerian coordinates. We will denote by $q = hu$ the discharge and we assume that the initial water depth $h(x, 0) = h_0(x)$ and velocity $u(x, 0) = u_0(x)$ are given.

The SWE are used as a paradigm of balance laws and can be written in the following condensed form

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})\partial_x z, \tag{1.2}$$

where $\mathbf{U} = (h, hu)^T$ corresponds to the conserved variables, the flux is given by $\mathbf{F}(\mathbf{U}) = (hu, hu^2 + gh^2/2)^T$ and $\mathbf{S}(\mathbf{U}) = (0, -gh)^T$ is the geometric source term.

Recall that the shallow water system is strictly hyperbolic over the phase space $\Omega = \{(h, hu)^T \in \mathbb{R}^2 \mid h > 0\}$ and that its eigenstructure is composed of two genuinely nonlinear fields associated with the eigenvalues $u - c$ and $u + c$, where $c = \sqrt{gh}$ is the sound speed. We also recall that the regions where $u^2 < c^2$ (resp. $u^2 > c^2$) are called subcritical (resp. supercritical).

When solving SWE one should take special care with the solutions corresponding to steady states

$$q = \text{constant}, \quad \frac{q^2}{2h^2} + g(h + z) = \text{constant}.$$

Among them, in this work we are interested in the so-called water at rest steady states, which are solutions given by

$$u = 0, \quad h + z = \text{constant}.$$

Numerical schemes that preserve water at rest solutions (respectively all steady states) are called well-balanced (respectively fully well-balanced). It is worth distinguishing between exactly preserving the steady states or a discrete approximation of them. The former are called exactly well-balanced while the latter are simply well-balanced schemes (see Gómez-Bueno et al. [1]).

Many different well-balanced schemes have been proposed in literature. We may cite [2–10] among many others.

Nevertheless, when dealing with low Froude number situations, the restriction on the time step due to the CFL condition makes such schemes inefficient. Indeed, in such situations a large final time will be needed and performing small time steps will require a lot of iterations. To overcome this difficulty, implicit or implicit-explicit schemes allow the use of a larger time step and therefore less time iterations are needed. In [11–14] implicit-explicit methods are proposed for three three dimensional shallow water flows. The extension into the DG framework to obtain high-order is described in Dumbser and Casulli [15]. The technique is also extended for bedload sediment transport in Garres-Díaz et al. [16]. Another possibility is to use implicit-explicit schemes (IMEX) which have successfully been applied to hyperbolic systems (see for instance [17,18]). In [9] an implicit-explicit scheme is proposed for the shallow water flows in the low Froude number limit.

In this work we propose the use of the so-called Lagrange-Projection decomposition in order to construct exactly well-balanced implicit finite volume schemes for system (1.1). A Lagrange-Projection type scheme is a two-step algorithm in which the system is first solved in Lagrangian coordinates and the results are then projected into Eulerian coordinates. The first step is usually called the Lagrangian step and the second one is referred to as the Projection or transport step. The main purpose of going back to Eulerian coordinates is that using pure Lagrangian coordinates and keeping track of moving meshes may be cumbersome and many complex situations may arise for the configuration of the moving cells, especially thinking on the extension to 2D. Moreover, this strategy allows us to decouple the acoustic and transport phenomena related to our equations and to design implicit-explicit and large time step schemes in a natural way. Indeed it is very useful for approximating subsonic or low Froude number flows, where the usual CFL time step driven by the acoustic waves can be very restrictive. Using the implicit or implicit-explicit schemes means that the CFL restriction reduces only to the slow transport step rather the more restrictive acoustic one. We address the reader to Chalons et al. [19–22] for further details.

The strategy proposed here follows the line of Castro et al. [23], where an explicit fully well-balanced finite volume method is proposed in the Lagrange-Projection framework. In that case, only the explicit case was studied and its extension to an implicit scheme was hinted as future work. That strategy was then extended in Morales de Luna et al. [24] by proposing an explicit high-order Lagrange-Projection scheme, but again for the explicit case. Those two papers set the basis for this work, where we intend to describe implicit schemes based on such approach. Let us recall that in Chalons et al. [19,21,22] one can find implicit-explicit schemes in the Lagrangian framework, where the source term is always treated explicitly. Those schemes where only first order accurate. One of the objectives here will be to extend the technique to second order implicit and implicit-explicit well-balanced schemes. Moreover, we set the basis for their extension to higher order and fully well-balanced schemes.

This paper is organized as follows: in Section 2 the main ideas concerning the Lagrange-Projection techniques are introduced. Then, in Section 3 two new second order numerical schemes are proposed for the Lagrangian step based on an implicit or implicit-explicit approach. The schemes are exactly well-balanced for water at rest steady states. Next, in Section 4 the projection step is described. Finally, some numerical simulations are shown in Section 5 in order to study the accuracy and efficiency of the new schemes. For the sake of completeness, we include the description of the explicit scheme in Appendix A.

2. The Lagrange-projection strategy

We consider the so-called Lagrangian coordinates, which follow the trajectories of the particles within the flow. In particular, consider any "fluid particle", ξ and define the characteristic curves

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi,t) = u(x(\xi,t),t), \\ x(\xi,0) = \xi. \end{cases} \tag{2.1}$$

Now, consider any function defined in Eulerian coordinates $(x,t) \mapsto \mathbf{U}(x,t)$. We then define by

$$\overline{\mathbf{U}}(\xi,t) = \mathbf{U}(x(\xi,t),t)$$

its equivalent in the Lagrangian variables (ξ,t).

Moreover, we define

$$L(\xi,t) = \frac{\partial x}{\partial \xi}(\xi,t),$$

the Jacobian of the Lagrangian map, which satisfies

$$\begin{cases} \partial_t L(\xi,t) = \partial_\xi \overline{u}(\xi,t), \\ L(\xi,0) = 1, \end{cases}$$

and for all $\mathbf{U}$ we have

$$\partial_\xi \overline{\mathbf{U}} = L\overline{\partial_x \mathbf{U}}$$

and

$$\partial_t \overline{\mathbf{U}} = \overline{\partial_t \mathbf{U} + u\partial_x \mathbf{U}}.$$

Since the SWE in Eulerian coordinates can be written for smooth solutions as

$$\begin{cases} \partial_t h + u\partial_x h + h\partial_x u = 0, \\ \partial_t (hu) + u\partial_x (hu) + hu\partial_x u + \partial_x \pi = -gh\partial_x z, \end{cases}$$

where $\pi = gh^2/2$, after multiplying both equations by $L(\xi,t)$ and using the previous identities for the derivatives of $\overline{\mathbf{U}}$ we obtain

$$\begin{cases} L\partial_t \overline{h} + \overline{h}\partial_\xi \overline{u} = 0, \\ L\partial_t (\overline{hu}) + \overline{hu}\partial_\xi \overline{u} + \partial_\xi \overline{\pi} + g\overline{h}\partial_\xi \overline{z} = 0, \end{cases}$$

and so

$$\begin{cases} \partial_t (L\overline{h}) = 0, \\ \partial_t (L\overline{hu}) + \partial_\xi \overline{\pi} + g\overline{h}\partial_\xi \overline{z} = 0. \end{cases} \tag{2.2}$$

Therefore, system (2.2) corresponds to the shallow water system expressed in Lagrangian coordinates.

From the numerical point of view, it will be useful to consider a relaxation approach of the Lagrangian system (2.2) (see Chalons et al. [22], Castro et al. [23]):

$$\begin{cases} \partial_t \tau - \tau_0 \partial_\xi u = 0, \\ \partial_t u + \tau_0 \partial_\xi \Pi + g\tau_0 h\partial_\xi z = \lambda(\pi - \Pi), \\ \partial_t \Pi + a^2 \tau_0 \partial_\xi u = 0, \end{cases} \tag{2.3}$$

where $\tau = 1/h$, $\tau_0 = \tau_{|_{t=0}}$, a is a constant satisfying the subcharacteristic condition $a > h\sqrt{gh}$ and $\lambda \to \infty$.

Note that in the relaxation system we have omitted the $\overline{(\cdot)}$ notation for the sake of simplicity.

From the numerical point of view, first (2.3) is solved with $\lambda = 0$, that is,

$$\begin{cases} \partial_t \tau - \tau_0 \partial_\xi u = 0, \\ \partial_t u + \tau_0 \partial_\xi \Pi + g\tau_0 h\partial_\xi z = 0, \\ \partial_t \Pi + a^2 \tau_0 \partial_\xi u = 0, \end{cases} \tag{2.4}$$

and then we set $\Pi = \pi(\tau)$ before the next iteration.

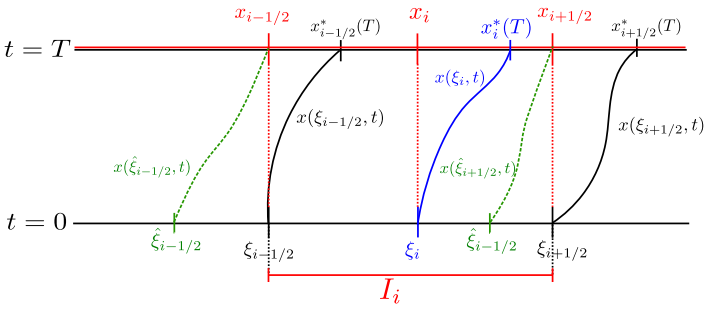


Fig. 1. Sketch of the relation between Eulerian and Lagrangian coordinates.

Now, defining two new variables $\overrightarrow{w} = \Pi + au$ and $\overleftarrow{w} = \Pi - au$, system (2.4) can be written as

$$\begin{cases} \partial_t \tau - \tau_0 \partial_\xi u = 0, \\ \partial_t \overrightarrow{w} + a\tau_0 \partial_\xi \overrightarrow{w} = -ga\tau_0 h \partial_\xi z, \\ \partial_t \overleftarrow{w} - a\tau_0 \partial_\xi \overleftarrow{w} = ga\tau_0 h \partial_\xi z. \end{cases} \tag{2.5}$$

The introduction of these two new variables allows one to obtain a system in which the second and third equations are just simple transport equations with a geometric source term.

Moreover, we can easily recover π and u from $\overrightarrow{w}$ and $\overleftarrow{w}$:

$$\pi = \frac{\overrightarrow{w} + \overleftarrow{w}}{2}, \qquad u = \frac{\overrightarrow{w} - \overleftarrow{w}}{2a}. \tag{2.6}$$

2.1. The Lagrange-projection numerical algorithm

In order to state the Lagrange-Projection numerical scheme, we shall proceed as it is usually done for finite volume schemes, although we distinguish here between Lagrangian and Eulerian coordinates. Space (in Lagrangian framework) will be discretized by means of a fixed space step $\Delta\xi$. The time step is denoted by Δt. The cells $I_i = [\xi_{i-1/2}, \xi_{i+1/2}]$ for $i \in \mathbb{Z}$ are then considered and we define the set of times $t^n = n\Delta t$ for $n \in \mathbb{N}$. In the Eulerian framework, the same step size, $\Delta x = \Delta\xi$, will be used. Similarly $x_{i+1/2} = i\Delta x$ and $x_i = (x_{i-1/2} + x_{i+1/2})/2$ correspond to the cell interfaces and the cell centers respectively and $I_i = [x_{i-1/2}, x_{i+1/2}]$. The Eulerian and Lagrangian space discretization will be then related using the following notation (see Fig. 1):

$$x^*_i(t) = x(\xi_i, t), \quad x^*_{i+1/2}(t) = x(\xi_{i+1/2}, t).$$

Remark that at time $t = 0$ we then have $x^*_i(0) = \xi_i = x_i$ and $x^*_{i+1/2}(0) = \xi_{i+1/2} = x_{i+1/2}$.

Let $\mathbf{U} = (h, hu)^T$. Assume that at the initial time $x \mapsto \mathbf{U}^0(x)$ is given. Then, define the discrete initial data $\mathbf{U}^0_i$, where

$$\mathbf{U}^0_i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}^0(x) dx, \quad i \in \mathbb{Z}.$$

The objective is to compute the values

$$\mathbf{U}^n_i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x, t^n) dx,$$

where $x \mapsto \mathbf{U}(x, t^n)$ corresponds to the solution of (1.1) at time t^n, $n \in \mathbb{N}$.

Starting from the sequence $\{\mathbf{U}^n_i\}_i$, it will be necessary to define the sequence $\{\mathbf{U}^{n+1}_i\}_i$, $n \in \mathbb{N}$, since $\{\mathbf{U}^0_i\}_i$ is known.

With this notation, the Lagrange-Projection algorithm will be described: from a given discrete sequence $\mathbf{U}^n_i = (h, hu)^n_i$, $i \in \mathbb{Z}$, corresponding to the approximations at time t^n, we have to compute $\mathbf{U}^{n+1}_i = (h, hu)^{n+1}_i$ at the next time. This is done in a two-step process:

1. Lagrangian step: update $\mathbf{U}^n_i$ to $\overline{\mathbf{U}}^{n+1}_i$ by numerically solving (2.2);
2. Projection step: update $\overline{\mathbf{U}}^{n+1}_i$ to $\mathbf{U}^{n+1}_i$ by projecting back to Eulerian framework.

3. The Lagrangian step

At this step we need to solve the shallow water system in Lagrangian coordinates (2.2). Although the aim here is to propose an implicit approximation of this system, we shall introduce, for the sake of completeness, an explicit approximation

as well in Appendix A. We will present first and second order schemes. For the implicit case, we will propose two versions of the scheme: nonlinear implicit and implicit-explicit.

Let us start by working on the second equation of system (2.2). The last term of this equation may be rewritten as

$$g\bar{h}\partial_\xi \bar{z} = gL\bar{h}\,\overline{\partial_x z} = gh_0\overline{\partial_x z}, \tag{3.1}$$

where we have used the first equation in (2.2) and denoted $h_0 = h_{|_{t=0}}$. Now, since z does not depend on time,

$$\overline{\partial_x z}(\xi_i, t) = \partial_x z(x(\xi_i, t)) = z'(x_i(t)),$$

where we have used the notation z' for $\partial_x z$. Therefore, the second equation in (2.2) is equivalent to

$$\partial_t(L\overline{hu}) + \partial_\xi \overline{\pi} + gh_0 z'(x_i(t)) = 0.$$

Now, integrating (2.2) in the interval $I_i = [\xi_{i-1/2}, \xi_{i+1/2}]$ we have:

$$\begin{cases} (L\bar{h})_i'(t) = 0, \\ (L\overline{hu})_i'(t) = -\dfrac{1}{\Delta\xi}\left(\pi^*_{i+1/2}(t) - \pi^*_{i-1/2}(t)\right) - \dfrac{1}{\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}} gP_{i,h_0}(\xi)z'(x(\xi,t))d\xi, \end{cases} \tag{3.2}$$

where we set $\bar{h}_i(0) = h_i^n$, and $(\overline{hu})_i(0) = (hu)_i^n$ as initial conditions. In this system, $\pi^*_{i\pm\frac{1}{2}}(t) \approx \overline{\pi}(\xi_{i\pm\frac{1}{2}}, t)$ and $P_{i,h_0}(\xi)$ is a reconstruction operator obtained from the sequence of cell values $\{h_i^n\}$:

$$P_{i,h_0}(\xi) = P_{i,h_0}(\xi; \{h_j^n\}_{j\in\mathcal{S}_i}),$$

being $\mathcal{S}_i$ the set of indexes belonging to the stencil corresponding to the ith cell.

Now we will consider a semi-discretization of system (2.5), that will play an important role in the definition of the discretization of the Lagrangian formulation of the SWE.

Using (3.1), system (2.5) can also be discretized in space using a first or second order scheme as follows:

$$\begin{cases} \tau_i'(t) = \dfrac{1}{h_{0,i}\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}} P_{i,u}(\xi,t)d\xi, \\ \overrightarrow{w}_i'(t) = -\dfrac{a}{h_{0,i}\Delta\xi}\left(\overrightarrow{\mathcal{W}}_{i+1/2}(t) - \overrightarrow{\mathcal{W}}_{i-1/2}(t)\right) - \dfrac{ga}{\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}} z'(x(\xi,t))d\xi, \\ \overleftarrow{w}_i'(t) = \dfrac{a}{h_{0,i}\Delta\xi}\left(\overleftarrow{\mathcal{W}}_{i+1/2}(t) - \overleftarrow{\mathcal{W}}_{i-1/2}(t)\right) + \dfrac{ga}{\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}} z'(x(\xi,t))d\xi, \end{cases} \tag{3.3}$$

with initial conditions $\tau_i(0) = \dfrac{1}{h_{0,i}}$, $\overrightarrow{w}_i(0) = \dfrac{1}{2}gh_{0,i}^2 + au_{0,i}$ and $\overleftarrow{w}_i(0) = \dfrac{1}{2}gh_{0,i}^2 - au_{0,i}$. $P_{i,u}(\xi,t)$ denotes the reconstruction operators of u defined from the cell values $\{u_i(t)\}$ on a given stencil. Finally, $\overrightarrow{\mathcal{W}}_{i+1/2}(t)$ and $\overleftarrow{\mathcal{W}}_{i+1/2}(t)$ are the numerical fluxes of the second and third equations of system (2.5), for which we will consider the upwind numerical fluxes given by:

$$\overrightarrow{\mathcal{W}}_{i+1/2}(t) = P_{i,\overrightarrow{w}}(\xi_{i+1/2}, t) = \overrightarrow{w}^t_{i+1/2-},$$
$$\overleftarrow{\mathcal{W}}_{i+1/2}(t) = P_{i+1,\overleftarrow{w}}(\xi_{i+1/2}, t) = \overleftarrow{w}^t_{i+1/2+},$$

where $P_{i,\overrightarrow{w}}$ and $P_{i,\overleftarrow{w}}$ are their respective reconstruction operators.

Finally, making use of the relations (2.6), we can define the following numerical fluxes for π and u at the interfaces:

$$\begin{aligned} \pi^*_{i+1/2}(t) &= \frac{P_{i,\overrightarrow{w}}(\xi_{i+1/2},t) + P_{i+1,\overleftarrow{w}}(\xi_{i+1/2},t)}{2}, \\ u^*_{i+1/2}(t) &= \frac{P_{i,\overrightarrow{w}}(\xi_{i+1/2},t) - P_{i+1,\overleftarrow{w}}(\xi_{i+1/2},t)}{2a}. \end{aligned} \tag{3.4}$$

As an example, let us now consider a first order in space and time implicit finite volume scheme for the SWE in Lagrangian coordinates that uses the previous formalism. In that case, the system is written as

$$\begin{aligned} \overrightarrow{w}_i^{n+1} &= \overrightarrow{w}_i^n - \frac{a\Delta t}{h_i^n\Delta\xi}\left(\overrightarrow{w}^{n+1}_{i+1/2-} - \overrightarrow{w}^{n+1}_{i-1/2-}\right) - \frac{a\Delta t}{\Delta\xi}gz'\left(x_i^{*,n+1}\right), \\ \overleftarrow{w}_i^{n+1} &= \overleftarrow{w}_i^n + \frac{a\Delta t}{h_i^n\Delta\xi}\left(\overleftarrow{w}^{n+1}_{i+1/2+} - \overleftarrow{w}^{n+1}_{i-1/2+}\right) + \frac{a\Delta t}{\Delta\xi}gz'\left(x_i^{*,n+1}\right), \end{aligned} \tag{3.5}$$

where we set $\overrightarrow{w}^{n+1}_{i+1/2-} = \overrightarrow{w}^{n+1}_i$ and $\overleftarrow{w}^{n+1}_{i+1/2+} = \overleftarrow{w}^{n+1}_{i+1}$. That is, the reconstruction operators reduce to the standard cell values at time t^{n+1}.

It should be noted that the presence of the source term requires the evaluation of z' at a point that we have denoted as $x_i^{*,n+1}$, which corresponds to a first order approximation of $x(\xi_i, t^{n+1})$. Here we propose the following approximation

$$x_i^{*,n+1} = \xi_i + \Delta t u_i^{*,n+1}, \tag{3.6}$$

where

$$u_i^{*,n+1} = \frac{u_{i-1/2}^{*,n+1} + u_{i+1/2}^{*,n+1}}{2} \tag{3.7}$$

and $u_{i\pm1/2}^{*,n+1}$ can be defined using the relations (3.4) as follows:

$$u_{i+1/2}^{*,n+1} = \frac{\overrightarrow{w}_i^{n+1} - \overleftarrow{w}_{i+1}^{n+1}}{2a}. \tag{3.8}$$

Note that (3.5)–(3.8) define a coupled non-linear system to be solved. Here we use a simple fixed-point algorithm where we first solve (3.5), fixing the value $x_i^{*,n+1}$ and then we update it by computing the velocity at the interfaces using (3.8). Observe that (3.5) reduces to two uncoupled linear systems that are simple to solve when $x_i^{*,n+1}$ is fixed.

Once $\overrightarrow{w}_i^{n+1}$ and $\overleftarrow{w}_i^{n+1}$ are computed, $(L\overline{hu})_i^{n+1}$ would be obtained as follows:

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \frac{\Delta t}{\Delta\xi}\left(\pi_{i+1/2}^{*,n+1} - \pi_{i-1/2}^{*,n+1}\right) - g\Delta t h_i^n z'\left(x_i^{*,n+1}\right), \tag{3.9}$$

where $\pi_{i+1/2}^{*,n+1}$ is computed using the relations (3.4), which reduce to

$$\pi_{i+1/2}^{*,n+1} = \frac{\overrightarrow{w}_i^{n+1} + \overleftarrow{w}_{i+1}^{n+1}}{2} \tag{3.10}$$

for a first order numerical scheme. Finally, we define

$$L_i^{n+1} = 1 + \frac{\Delta t}{\Delta\xi}\left(u_{i+1/2}^{*,n+1} - u_{i-1/2}^{*,n+1}\right). \tag{3.11}$$

Notice that we could avoid the solution of the non-linear system (3.5)–(3.8) by considering the following explicit-implicit first order numerical scheme

$$\begin{aligned}
\overrightarrow{w}_i^{n+1} &= \overrightarrow{w}_i^n - \frac{a\Delta t}{h_i^n \Delta\xi}\left(\overrightarrow{w}_{i+1/2-}^{n+1} - \overrightarrow{w}_{i-1/2-}^{n+1}\right) - \frac{a\Delta t}{\Delta\xi} g z'\left(x_i^n\right), \\
\overleftarrow{w}_i^{n+1} &= \overleftarrow{w}_i^n + \frac{a\Delta t}{h_i^n \Delta\xi}\left(\overleftarrow{w}_{i+1/2+}^{n+1} - \overleftarrow{w}_{i-1/2+}^{n+1}\right) + \frac{a\Delta t}{\Delta\xi} g z'\left(x_i^n\right).
\end{aligned} \tag{3.12}$$

Once $\overleftarrow{w}_i^{n+1}$ and $\overrightarrow{w}_i^{n+1}$ are computed, then $(L\overline{hu})_i^{n+1}$ is obtained as follows:

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \frac{\Delta t}{\Delta\xi}\left(\pi_{i+1/2}^{*,n+1} - \pi_{i-1/2}^{*,n+1}\right) - g\Delta t h_i^n z'\left(x_i^n\right). \tag{3.13}$$

Finally, L_i^{n+1} is updated using (3.11).

Fully explicit versions of the previous numerical scheme are straightforward and their extension to second order is also possible (see Appendix A).

Let us remark that, in order to ensure stability, the parameter a must be chosen sufficiently large according to the so-called subcharacteristic condition $a > h\sqrt{gh}$ (see Chalons et al. [19,21,22]). Moreover, the explicit Lagrangian step is stable provided the following CFL condition is satisfied

$$a\Delta t \leq \frac{1}{2} h \Delta\xi. \tag{3.14}$$

The use of an implicit approach for the Lagrangian step makes that condition (3.14) is no longer required.

Unfortunately, neither the numerical scheme defined by (3.5)–(3.9) or (3.12), (3.13) are well-balanced for the water at rest solution. In order to define well-balanced numerical schemes for the water at rest solution we follow the procedure described in Castro and Parés [25]. More explicitly, at every cell $I_i = [\xi_{i-1/2}, \xi_{i+1/2}]$ we look for a steady state that matches with the given cell average. As we are interested in first and second order schemes and we focus on water at rest steady states, this condition reduces to defining an in-cell stationary water height $h_i^{e,n}(\xi)$ such that $h_i^{e,n}(\xi_i) = h_i^n$. Taking into account the special expression of the water at rest solutions for the SWE, $h_i^{e,n}(\xi)$ is given by

$$h_i^{e,n}(\xi) = h_i^n + z(\xi_i) - z(\xi). \tag{3.15}$$

We also define $\pi_i^{e,n}$ as

$$\pi_i^{e,n}(\xi) = \frac{1}{2} g\left(h_i^{e,n}(\xi)\right)^2. \tag{3.16}$$

Notice that $h_i^{e,n}(\xi)$ exactly satisfies

$$\partial_\xi \overline{\pi_i^{e,n}} + g\overline{Lh_i^{e,n}z'} = 0. \tag{3.17}$$

Now, the second equation of (3.2) could be rewritten equivalently as follows:

$$\begin{aligned}(\overline{Lhu})_i'(t) = &-\frac{1}{\Delta\xi}\left(\pi^*_{i+1/2}(t) - \pi^*_{i-1/2}(t)\right) + \frac{1}{\Delta\xi}\left(\pi_i^{e,n}(x_{i+1/2}(t)) - \pi_i^{e,n}(x_{i-1/2}(t))\right)\\ &-\frac{1}{\Delta\xi}\int_{\xi_{i-1/2}}^{\xi_{i+1/2}} g\left(P_{i,h_0}(\xi) - L(\xi,t)h_i^{e,n}(x(\xi,t))\right)z'(x(\xi,t))d\xi,\end{aligned} \tag{3.18}$$

where $x_{i+1/2}(t) = x(\xi_{i+1/2}, t)$.

Proceeding analogously with the second and third equations of (3.3), one has:

$$\begin{cases}\overrightarrow{w}_i'(t) = -\dfrac{a}{h_{0,i}\Delta\xi}\left(\overrightarrow{\mathcal{W}}_{i+1/2}(t) - \overrightarrow{\mathcal{W}}_{i-1/2}(t)\right) + \dfrac{a}{h_{0,i}\Delta\xi}\left(\pi_i^{e,n}(x_{i+1/2}(t)) - \pi_i^{e,n}(x_{i-1/2}(t))\right)\\ \qquad -\dfrac{ga}{\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}}\left(1 - \frac{L(\xi,t)h_i^{e,n}(x(\xi,t))}{P_{i,h_0}(\xi)}\right)z'(x(\xi,t))d\xi,\\ \overleftarrow{w}_i'(t) = \dfrac{a}{h_{0,i}\Delta\xi}\left(\overleftarrow{\mathcal{W}}_{i+1/2}(t) - \overleftarrow{\mathcal{W}}_{i-1/2}(t)\right) - \dfrac{a}{h_{0,i}\Delta\xi}\left(\pi_i^{e,n}(x_{i+1/2}(t)) - \pi_i^{e,n}(x_{i-1/2}(t))\right)\\ \qquad +\dfrac{ga}{\Delta\xi}\displaystyle\int_{\xi_{i-1/2}}^{\xi_{i+1/2}}\left(1 - \frac{L(\xi,t)h_i^{e,n}(x(\xi,t))}{P_{i,h_0}(\xi)}\right)z'(x(\xi,t))d\xi.\end{cases} \tag{3.19}$$

Note that the semi-discrete scheme described in (3.18) and (3.19) is equivalent to (3.2) and (3.3). Now, we need to describe the procedure to define the reconstructions operators of the unknowns that are exactly well-balanced in the sense defined in Gómez-Bueno et al. [1].

3.1. Exactly well-balanced reconstruction operators

In the next paragraphs we describe the reconstruction procedure that we define in order to achieve a numerical scheme that is exactly well-balanced for the water at rest solutions.

Let us suppose that $\{h_i^n\}$ and $\{(hu)_i^n\}$ are known. Then the reconstruction operators for P_{i,h_0}, $P^n_{i,\overleftarrow{w}}$ and $P^n_{i,\overrightarrow{w}}$ are defined as follows:

$$P_{i,h_0}(\xi) = h_i^{e,n}(\xi) + Q_i^n(\xi; \{h_j^{n,f}\}_{j\in\mathcal{S}_i}), \tag{3.20}$$

where $h_i^{e,n}(\xi)$ is given in (3.15), $h_j^{n,f} = h_j^n - h_i^{e,n}(\xi_j)$, $j \in \mathcal{S}_i$, that is $h_j^{n,f}$ are the fluctuations with respect to the steady state at the stencil, and $Q_i^n(\xi)$ is a standard first or second order reconstruction operator. Note that as we are only interested in first and second order numerical methods, the fluctuations are computed using punctual evaluations at the cell centers. If order higher than two is required, then cell averages computed with a quadrature formula should be used.

Now, $P^n_{i,\overrightarrow{w}}$ is defined equally as

$$P^n_{i,\overrightarrow{w}}(\xi) = \pi_i^{e,n}(\xi) + Q_i^n(\xi; \{\overrightarrow{w}_j^{n,f}\}_{j\in\mathcal{S}_i}), \tag{3.21}$$

where $\overrightarrow{w}_j^{n,f}$ is given by $\overrightarrow{w}_j^{n,f} = \overrightarrow{w}_j^n - \pi_i^{e,n}(\xi_j)$, $j \in \mathcal{S}_i$, that is, they are again the fluctuations with respect to the local steady state at the stencil. $P^n_{i,\overleftarrow{w}}$ is defined analogously.

Finally, $P^n_{i,u}$ is a standard first or second order reconstruction operator computed with the cell values $\{u_i^n\}$. Note that as we are only interested in water at rest steady state, no special care has to be taken for the reconstruction operator of u or hu. This would not be the case for moving equilibria. Moreover, if order higher than two is required, then a more sophisticated procedure should be used as the cell averages of u cannot be obtained by simply setting $u_i^n = \dfrac{(hu)_i^n}{h_i^n}$.

Observe that P_{i,h_0}, $P^n_{i,\overleftarrow{w}}$, $P^n_{i,\overrightarrow{w}}$ and $P^n_{i,u}$ are exactly well-balanced operators in the sense that if $\{h_i^n\}$ and $\{(hu)_i^n\}$ are the cell averages of a given water at rest steady state that are computed with the mid-point rule, that is $h_i^n = h^e(\xi_i)$, $hu_i^n = 0$, then

$$P_{i,h_0}(\xi) = h^e(\xi)_{|_{I_i}},\ P^n_{i,\overleftarrow{w}}(\xi) = P^n_{i,\overrightarrow{w}}(\xi) = \pi^e(\xi)_{|_{I_i}},\ \text{and } P^n_{i,u}(\xi) = 0.$$

Given our interest in the definition of implicit solvers, we also need to provide a reconstruction procedure for any value $t \in [t^n, t^{n+1}]$ that is able to preserve water at rest steady states. In particular we need to define $P_{i,\overleftarrow{w}}(\xi,t)$ and $P_{i,\overrightarrow{w}}(\xi,t)$. Let us define $P_{i,\overrightarrow{w}}(\xi,t)$ and the definition of $P_{i,\overleftarrow{w}}(\xi,t)$ will be analogous. $P_{i,\overrightarrow{w}}(\xi,t)$ will be defined using the exactly well-balanced reconstruction operator at time $t = t^n$, $P^n_{i,\overrightarrow{w}}$, described previously, and a standard reconstruction operator for the time fluctuations, that is:

$$P_{i,\overrightarrow{w}}(\xi,t) = P^n_{i,\overrightarrow{w}}(\xi) + \widetilde{Q}_i(\xi,t), \tag{3.22}$$

where $\widetilde{Q}_i(\xi,t)$ is a standard reconstruction operator defined in terms of the time fluctuations, that is

$$\widetilde{Q}_i(\xi,t) = \widetilde{Q}_i(\xi;\{\overrightarrow{w}_j^{t,f}\}_{j\in\mathcal{S}_i}), \text{ where } \overrightarrow{w}_j^{t,f} = \overrightarrow{w}_j(t) - \overrightarrow{w}_j^n, \ j\in\mathcal{S}_i.$$

Let us show a first order reconstruction operator for $\overrightarrow{w}$ at cell I_i for any value $t\in[t^n,t^{n+1}]$

$$P_{i,\overrightarrow{w}}^{o1}(\xi,t) = P_{i,\overrightarrow{w}}^{n,o1}(\xi) + \widetilde{Q}_i^{o1}(\xi,t),$$

where $\widetilde{Q}_i^{o1}(\xi,t)$ is the standard first order reconstruction operator at cell I_i, that is, $\widetilde{Q}_i^{o1}(\xi,t) = \overrightarrow{w}_i^{t,f} = \overrightarrow{w}_i(t) - \overrightarrow{w}_i^n$. Now, taking into account the definition of $P_{i,\overrightarrow{w}}^n(\xi)$, we have that

$$P_{i,\overrightarrow{w}}^{o1}(\xi,t) = \pi_i^{e,n}(\xi) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i(t). \tag{3.23}$$

The second order reconstruction operator is defined in the same way, that is

$$P_{i,\overrightarrow{w}}^{o2}(\xi,t) = P_{i,\overrightarrow{w}}^{n,o2}(\xi) + \widetilde{Q}_i^{o2}(\xi,t), \tag{3.24}$$

where $\widetilde{Q}_i^{o2}(\xi,t)$ is a standard second order reconstruction operator defined in terms of the time fluctuations $\{\overrightarrow{w}_i^{t,f}\}_{j\in\mathcal{S}_i}$. Now, taking into account the definition of $P_{i,\overrightarrow{w}}^{n,o2}$ we have that

$$P_{i,\overrightarrow{w}}^{o2}(\xi,t) = \pi_i^{e,n}(\xi) + Q_i^{n,o2}(\xi;\{\overrightarrow{w}_j^{n,f}\}_{j\in\mathcal{S}_i}) + \widetilde{Q}_i^{o2}(\xi,t;\{\overrightarrow{w}_j^{t,f}\}_{j\in\mathcal{S}_i}),$$

where $\overrightarrow{w}_j^{n,f} = \overrightarrow{w}_j^n - \pi_i^{e,n}(\xi_j)$. In the previous expression, $Q_i^{n,o2}$ and $\widetilde{Q}_i^{o2}$ are standard second order reconstruction operators. In this case, to avoid the non-linear dependency of the limiters at any time t, we consider here that both $Q_i^{n,o2}$ and $\widetilde{Q}_i^{o2}$ use the same limiters computed at time $t=t^n$.

In particular, $Q_i^{n,o2}(\xi)$ is defined as follows:

$$Q_i^{n,o2}(\xi) = \overrightarrow{w}_i^n - \pi_i^{e,n}(\xi_i) + \Delta^n\overrightarrow{w}_i^{n,f}(\xi-\xi_i), \tag{3.25}$$

where

$$\Delta^n\overrightarrow{w}_i^{n,f} = \frac{1}{\Delta\xi}\left(\phi_{i+}^n\left(\overrightarrow{w}_i^{n,f} - \overrightarrow{w}_{i-1}^{n,f}\right) + \phi_{i-}^n\left(\overrightarrow{w}_{i+1}^{n,f} - \overrightarrow{w}_i^{n,f}\right)\right), \tag{3.26}$$

with ϕ_{i-}^n and ϕ_{i+}^n slope limiters. Here we use the following:

$$\phi_{i-}^n = \begin{cases} \dfrac{|d_{i-}|}{|d_{i-}|+|d_{i+}|} & \text{if } |d_{i-}|+|d_{i+}|>0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{i+}^n = \begin{cases} \dfrac{|d_{i+}|}{|d_{i-}|+|d_{i+}|} & \text{if } |d_{i-}|+|d_{i+}|>0, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$d_{i-} = \overrightarrow{w}_i^{n,f} - \overrightarrow{w}_{i-1}^{n,f}, \quad d_{i+} = \overrightarrow{w}_{i+1}^{n,f} - \overrightarrow{w}_i^{n,f}.$$

$\widetilde{Q}_i^{o2}(\xi,t)$ is defined as follows

$$\widetilde{Q}_i^{o2}(\xi,t) = \overrightarrow{w}_i(t) - \overrightarrow{w}_i^n + \Delta^t\overrightarrow{w}_i^{t,f}(\xi-\xi_i),$$

where

$$\Delta^t\overrightarrow{w}_i^{t,f} = \frac{1}{\Delta\xi}\left(\widetilde{\phi}_{i-}^n\left(\overrightarrow{w}_{i+1}^{t,f} - \overrightarrow{w}_i^{t,f}\right) + \widetilde{\phi}_{i+}^n\left(\overrightarrow{w}_i^{t,f} - \overrightarrow{w}_{i-1}^{t,f}\right)\right),$$

with $\widetilde{\phi}_{i\pm}^n = \phi_{i\pm}^n$.

Taking into account the definitions of $Q_i^{n,o2}$ and $\widetilde{Q}_i^{t,o2}$, we obtain that

$$P_{i,\overrightarrow{w}}^{o2}(\xi,t) = \pi_i^{e,n}(\xi) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i(t) + \Delta^n\overrightarrow{w}_i^{n,f}(\xi-\xi_i) + \Delta^t\overrightarrow{w}_i^{t,f}(\xi-\xi_i). \tag{3.27}$$

3.2. Implicit and implicit-explicit exactly well-balanced Lagrangian schemes

In this subsection we are going to describe the implicit and implicit-explicit first and second order exactly well-balanced Lagrangian schemes. The first and second order explicit Lagrangian schemes are described in Appendix A. In the implicit and implicit-explicit case we use the mid-point rule for approximating the integrals in (3.18) and (3.19) and we obtain the following semi-discrete formulation:

$$(L\overline{hu})_i'(t) = -\mathcal{L}_i(t) - \mathcal{G}_i(t) \tag{3.28}$$

where

$$\mathcal{L}_i(t) = \frac{1}{\Delta\xi}\left(\pi^*_{i+1/2}(t) - \pi^*_{i-1/2}(t)\right) \tag{3.29}$$

$$\mathcal{G}_i(t) = -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(x_{i+1/2}(t)) - \pi_i^{e,n}(x_{i-1/2}(t))\right) + g\left(P_{i,h_0}(\xi_i) - L_i(t)h_i^{e,n}(x(\xi_i,t))\right)z'(x(\xi_i,t)) \tag{3.30}$$

and

$$\begin{cases} \overrightarrow{w}_i'(t) = -(\mathcal{L}_{\overrightarrow{w}})_i(t) - \dfrac{a}{h_{0,i}}\mathcal{G}_i(t), \\ \overleftarrow{w}_i'(t) = -(\mathcal{L}_{\overleftarrow{w}})_i(t) + \dfrac{a}{h_{0,i}}\mathcal{G}_i(t), \end{cases} \tag{3.31}$$

where

$$(\mathcal{L}_{\overrightarrow{w}})_i(t) = \frac{a}{h_{0,i}\Delta\xi}\left(\overrightarrow{\mathcal{W}}_{i+1/2}(t) - \overrightarrow{\mathcal{W}}_{i-1/2}(t)\right) \tag{3.32}$$

$$(\mathcal{L}_{\overleftarrow{w}})_i(t) = -\frac{a}{h_{0,i}\Delta\xi}\left(\overleftarrow{\mathcal{W}}_{i+1/2}(t) - \overleftarrow{\mathcal{W}}_{i-1/2}(t)\right), \tag{3.33}$$

where we recall that $x_{i\pm1/2}(t) = x(\xi_{i\pm1/2}, t)$.

Now, we are ready to start the definition of first and second order exactly well-balanced implicit and implicit-explicit schemes in Lagrangian coordinates. Here we suppose that $\{h_i^n\}$ and $\{(hu)_i^n\}$ are known and we integrate the system in the interval $[t^n, t^{n+1}]$ to compute the new states at t^{n+1}.

3.2.1. First order schemes

Implicit scheme

Firstly, let us consider the definition of the first order implicit scheme. In that case, the scheme reads as follows:

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \Delta t\left(\mathcal{L}_i^{n+1} + \mathcal{G}_i^{n+1}\right), \tag{3.34}$$

where

$$\mathcal{L}_i^{n+1} = \frac{1}{\Delta\xi}\left(\pi^{*,n+1}_{i+1/2} - \pi^{*,n+1}_{i-1/2}\right), \tag{3.35}$$

and

$$\mathcal{G}_i^{n+1} = -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(x^{*,n+1}_{i+1/2}) - \pi_i^{e,n}(x^{*,n+1}_{i-1/2})\right) + g\left(h_i^n - L_i^{n+1}h_i^{e,n}(x_i^{*,n+1})\right)z'(x_i^{*,n+1}), \tag{3.36}$$

with L_i^{n+1} defined as in (3.11), $x_i^{*,n+1}$ defined as in (3.6) and $x^{*,n+1}_{i\pm1/2}$ defined as

$$x^{*,n+1}_{i\pm1/2} = \xi_{i\pm1/2} + \Delta t u^{*,n+1}_{i\pm1/2}. \tag{3.37}$$

We recall that $\pi^{*,n+1}_{i\pm1/2}$ and $u^{*,n+1}_{i\pm1/2}$ are defined by (3.4), that is,

$$\pi^{*,n+1}_{i+1/2} = \frac{P^{o1}_{i,\overrightarrow{w}}(\xi_{i+1/2}, t^{n+1}) + P^{o1}_{i+1,\overleftarrow{w}}(\xi_{i+1/2}, t^{n+1})}{2} = \frac{\overrightarrow{w}^{n+1}_{i+1/2-} + \overleftarrow{w}^{n+1}_{i+1/2+}}{2} \tag{3.38}$$

and

$$u^{*,n+1}_{i+1/2} = \frac{P^{o1}_{i,\overrightarrow{w}}(\xi_{i+1/2}, t^{n+1}) - P^{o1}_{i+1,\overleftarrow{w}}(\xi_{i+1/2}, t^{n+1})}{2a} = \frac{\overrightarrow{w}^{n+1}_{i+1/2-} - \overleftarrow{w}^{n+1}_{i+1/2+}}{2a} \tag{3.39}$$

where

$$\begin{aligned} \overrightarrow{w}^{n+1}_{i+1/2-} &= P^{o1}_{i,\overrightarrow{w}}(\xi_{i+1/2}, t^{n+1}) = \pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i^{n+1}, \\ \overleftarrow{w}^{n+1}_{i+1/2+} &= P^{o1}_{i+1,\overleftarrow{w}}(\xi_{i+1/2}, t^{n+1}) = \pi_{i+1}^{e,n}(\xi_{i+1/2}) - \pi_{i+1}^{e,n}(\xi_i) + \overleftarrow{w}_{i+1}^{n+1}. \end{aligned} \tag{3.40}$$

Note that $\pi^{*,n+1}_{i\pm1/2}$ and $u^{*,n+1}_{i\pm1/2}$ are defined in terms of $\overrightarrow{w}^{n+1}$ and $\overleftarrow{w}^{n+1}$, that are the solutions of the non-linear system

$$\begin{cases} \overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^n - \Delta t\left((\mathcal{L}_{\overrightarrow{w}})_i^{n+1} + \dfrac{a}{h_i^n}\mathcal{G}_i^{n+1}\right), \\ \overleftarrow{w}_i^{n+1} = \overleftarrow{w}_i^n - \Delta t\left((\mathcal{L}_{\overleftarrow{w}})_i^{n+1} - \dfrac{a}{h_i^n}\mathcal{G}_i^{n+1}\right), \end{cases} \tag{3.41}$$

where

$$\begin{aligned} (\mathcal{L}_{\overrightarrow{w}})_i^{n+1} &= \frac{a}{h_i^n\Delta\xi}\left(\overrightarrow{w}^{n+1}_{i+1/2-} - \overrightarrow{w}^{n+1}_{i-1/2-}\right), \\ (\mathcal{L}_{\overleftarrow{w}})_i^{n+1} &= -\frac{a}{h_i^n\Delta\xi}\left(\overleftarrow{w}^{n+1}_{i+1/2+} - \overleftarrow{w}^{n+1}_{i-1/2+}\right). \end{aligned} \tag{3.42}$$

In order to apply the previous numerical scheme, we first solve the non-linear system (3.41), and then, (3.34) is updated using $\{\overrightarrow{w}_i^{n+1}\}$ and $\{\overleftarrow{w}_i^{n+1}\}$.

The non-linear system (3.41) is solved by means of a fixed-point iteration where $\{\overrightarrow{w}_i^n\}$ and $\{\overleftarrow{w}_i^n\}$ are given as initial guess. Here (3.41) is solved as follows: we fix the values of $x^{*,n+1}_{i\pm1/2}$ and $x_i^{*,n+1}$ in $(\mathcal{G}_w)_i^{n+1}$ and then we solve the two linear systems in (3.41). Next $u^{*,n+1}_{i\pm1/2}$ are updated as well as $x^{*,n+1}_{i\pm1/2}$ and $x_i^{*,n+1}$ with the new computed values of $\{\overleftarrow{w}_i^{n+1}\}$ and $\{\overrightarrow{w}_i^{n+1}\}$.

Theorem 1. *Given a water at rest stationary solution of system (1.1) with a continuous bottom topography z, the scheme defined by (3.34) and (3.41) is exactly well-balanced.*

Proof. Let $h_i^n = h^{e,n}(\xi_i)$, $\pi_i^n = \pi^{e,n}(\xi_i)$, $u_i^n = 0$. In the first iteration of the fixed-point algorithm, $x^{*,n+1}_{i\pm1/2} = \xi_{i\pm1/2}$, $x_i^{*,n+1} = \xi_i$. Let $\overrightarrow{w}_i^{n+1,l}$ be the result of the lth iteration of the fixed-point algorithm. Then,

$$\begin{aligned} \overrightarrow{w}_i^{n+1,1} &= \overrightarrow{w}_i^n - \frac{\Delta t a}{h_i^n\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i^{n+1,1} - \pi_{i-1}^{e,n}(\xi_{i-1/2}) + \pi_{i-1}^{e,n}(\xi_{i-1}) - \overrightarrow{w}_{i-1}^{n+1,1}\right) \\ &\quad + \frac{\Delta t a}{h_i^n\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right) - \frac{ga}{h_i^n}(h_i^n - L_i^{n+1}h_i^{e,n}(\xi_i))z'(\xi_i) \\ &= \overrightarrow{w}_i^n - \frac{\Delta t a}{h_i^n\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i^{n+1,1} - \pi_i^{e,n}(\xi_{i-1/2}) + \pi_{i-1}^{e,n}(\xi_{i-1}) - \overrightarrow{w}_{i-1}^{n+1,1}\right) \\ &\quad + \frac{\Delta t a}{h_i^n\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right) \\ &= \overrightarrow{w}_i^n - \frac{\Delta t a}{h_i^n\Delta\xi}\left(-\pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i^{n+1,1} + \pi_{i-1}(\xi_{i-1}) - \overrightarrow{w}_{i-1}^{n+1,1}\right), \end{aligned}$$

where we have used that $\pi_{i-1}^{e,n}(\xi_{i-1/2}) = \pi_i^{e,n}(\xi_{i-1/2})$ and that $L_i^{n+1} = 1$ since in the first iteration $u^{*,n+1}_{i\pm1/2} = 0$. Note that at this point we are using that the bottom topography is continuous what implies that h^e is continuous as well as π^e. This numerical scheme could be modified in order to be exactly well-balanced for non-smooth steady states (see Castro and Parés [25], Bouchut [26]).

It is clear that $\overrightarrow{w}_i^{n+1,1} = \pi_i^{e,n}(\xi_i)$, $\overrightarrow{w}_{i-1}^{n+1,1} = \pi_{i-1}^{e,n}(\xi_{i-1})$ is a solution of the previous equation and therefore, $\overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^n = \pi_i^{e,n}(\xi_i)$. The result for $\overleftarrow{w}$ is analogous.

Once we know $\overrightarrow{w}^{n+1}$ and $\overleftarrow{w}^{n+1}$, we need to compute $\pi^{*,n+1}_{i+1/2}$ and $\pi^{*,n+1}_{i+1/2}$ making use of (3.38), (3.39) and (3.23) and we obtain

$$\begin{aligned} \pi^{*,n+1}_{i+1/2} &= \frac{\pi_i^{e,n}(\xi_{i+1/2}) + \pi_{i+1}^{e,n}(\xi_{i+1/2})}{2} = \frac{\pi_i^{e,n}(\xi_{i+1/2}) + \pi_i^{e,n}(\xi_{i+1/2})}{2} = \pi_i^{e,n}(\xi_{i+1/2}), \\ u^{*,n+1}_{i+1/2} &= \frac{\pi_i^{e,n}(\xi_{i+1/2}) - \pi_{i+1}^{e,n}(\xi_{i+1/2})}{2a} = \frac{\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i+1/2})}{2a} = 0. \end{aligned}$$

Then, since $L_i^{n+1} = 1$ and $x^{*,n+1}_{i\pm1/2} = \xi_{i\pm1/2}$,

$$\begin{aligned} \mathcal{L}_i^{n+1} &= \frac{1}{\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right), \\ \mathcal{G}_i^{n+1} &= -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right) + g\left(h_i^n - h_i^{e,n}(\xi_i)\right)z'(\xi_i) \\ &= -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right). \end{aligned}$$

Finally, $(L\overline{h})_i^{n+1} = h_i^n$, $(L\overline{hu})_i^{n+1} = (hu)_i^n$ and, using the fact that $L_i^{n+1} = 1$, we get that the scheme is exactly well-balanced. □

Implicit-explicit scheme

We can also consider the following first order implicit-explicit scheme:

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \Delta t\left(\mathcal{L}_i^{n+1} + \mathcal{G}_i^n\right), \tag{3.43}$$

where $\mathcal{L}_i^{n+1}$ is defined as in (3.35) and

$$\mathcal{G}_i^n = -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_{i-1/2})\right) \tag{3.44}$$

since the second term of $\mathcal{G}$ evaluated at time t^n is zero.

In this case, $\overrightarrow{w}^{n+1}$ and $\overleftarrow{w}^{n+1}$ are the solutions of the following linear system

$$\begin{cases} \overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^n - \Delta t\left((\mathcal{L}_{\overrightarrow{w}})_i^{n+1} + \dfrac{a}{h_i^n}\mathcal{G}_i^n\right), \\ \overleftarrow{w}_i^{n+1} = \overleftarrow{w}_i^n - \Delta t\left((\mathcal{L}_{\overleftarrow{w}})_i^{n+1} - \dfrac{a}{h_i^n}\mathcal{G}_i^n\right) \end{cases} \tag{3.45}$$

with $(\mathcal{L}_{\overrightarrow{w}})_i^{n+1}$, $(\mathcal{L}_{\overleftarrow{w}})_i^{n+1}$ defined as in (3.42). Remark that now the solution of (3.45) is straightforward and the fixed point iterations are avoided, making this implicit-explicit version more efficient.

Theorem 2. *Given a water at rest stationary solution of system (1.1) with a continuous bottom topography z, the scheme defined by (3.43) and (3.45) is exactly well-balanced.*

Proof. The proof is analogous to the previous one. □

3.2.2. Second order schemes

Implicit scheme

Let us now define the second order well-balanced implicit scheme. In this case, applying the trapezoidal rule

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \frac{\Delta t}{2}\left(\mathcal{L}_i^n + \mathcal{L}_i^{n+1} + \mathcal{G}_i^n + \mathcal{G}_i^{n+1}\right), \tag{3.46}$$

where

$$\mathcal{L}_i^n = \frac{1}{\Delta\xi}\left(\pi_{i+1/2}^{*,n} - \pi_{i-1/2}^{*,n}\right),$$
$$\mathcal{L}_i^{n+1} = \frac{1}{\Delta\xi}\left(\pi_{i+1/2}^{*,n+1} - \pi_{i-1/2}^{*,n+1}\right),$$

and $\mathcal{G}_i^n$ is computed as in (3.44) and $\mathcal{G}_i^{n+1}$ as in (3.36) where now we use the following second order approximations

$$x_{i\pm1/2}^{*,n+1} = x_{i\pm1/2} + \frac{\Delta t}{2}\left(u_{i\pm1/2}^{*,n+1} + u_{i\pm1/2}^{*,n}\right),$$
$$x_i^{*,n+1} = x_i + \frac{\Delta t}{4}\left(u_{i+1/2}^{*,n+1} + u_{i-1/2}^{*,n+1} + u_{i+1/2}^{*,n} + u_{i-1/2}^{*,n}\right),$$
$$L_i^{n+1} = 1 + \frac{\Delta t}{2\Delta\xi}\left(u_{i+1/2}^{*,n+1} - u_{i-1/2}^{*,n+1} + u_{i+1/2}^{*,n} - u_{i-1/2}^{*,n}\right).$$

The values $\pi_{i+1/2}^{*,n}$, $\pi_{i+1/2}^{*,n+1}$, $u_{i+1/2}^{*,n}$, $u_{i+1/2}^{*,n+1}$ can be obtained as follows

$$\pi_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2}, t^n) + P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2}, t^n)}{2} = \frac{\overrightarrow{w}_{i+1/2-}^n + \overleftarrow{w}_{i+1/2+}^n}{2} \tag{3.47}$$

$$\pi_{i+1/2}^{*,n+1} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2}, t^{n+1}) + P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2}, t^{n+1})}{2} = \frac{\overrightarrow{w}_{i+1/2-}^{n+1} + \overleftarrow{w}_{i+1/2+}^{n+1}}{2} \tag{3.48}$$

$$u_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2}, t^n) - P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2}, t^n)}{2a} = \frac{\overrightarrow{w}_{i+1/2-}^n - \overleftarrow{w}_{i+1/2+}^n}{2a} \tag{3.49}$$

$$u_{i+1/2}^{*,n+1} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2}, t^{n+1}) - P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2}, t^{n+1})}{2a} = \frac{\overrightarrow{w}_{i+1/2-}^{n+1} - \overleftarrow{w}_{i+1/2+}^{n+1}}{2a} \tag{3.50}$$

where in order to compute $\overrightarrow{w}_{i+1/2}^n$, $\overrightarrow{w}_{i+1/2}^{n+1}$, $\overleftarrow{w}_{i+1/2}^n$ and $\overleftarrow{w}_{i+1/2}^{n+1}$ we use the reconstruction proposed in (3.27).

Similarly, $\overrightarrow{w}^{n+1}$ and $\overleftarrow{w}^{n+1}$ are the solutions of the following non-linear system

$$\begin{cases} \overrightarrow{w}_i^{n+1} = \overrightarrow{w}_i^n - \dfrac{\Delta t}{2}\left((\mathcal{L}_{\overrightarrow{w}})_i^n + (\mathcal{L}_{\overrightarrow{w}})_i^{n+1} + \dfrac{a}{h_i^n}(\mathcal{G}_i^n + \mathcal{G}_i^{n+1}) \right), \\ \overleftarrow{w}_i^{n+1} = \overleftarrow{w}_i^n - \dfrac{\Delta t}{2}\left((\mathcal{L}_{\overleftarrow{w}})_i^n + (\mathcal{L}_{\overleftarrow{w}})_i^{n+1} - \dfrac{a}{h_i^n}(\mathcal{G}_i^n + \mathcal{G}_i^{n+1}) \right) \end{cases} \tag{3.51}$$

where in this case

$$\begin{aligned} (\mathcal{L}_{\overrightarrow{w}})_i^n &= \frac{a}{2h_i^n \Delta\xi}\left(\overrightarrow{w}_{i+1/2-}^n - \overrightarrow{w}_{i-1/2-}^n\right), \\ (\mathcal{L}_{\overleftarrow{w}})_i^n &= -\frac{a}{2h_i^n \Delta\xi}\left(\overleftarrow{w}_{i+1/2+}^n - \overleftarrow{w}_{i-1/2+}^n\right). \\ (\mathcal{L}_{\overrightarrow{w}})_i^{n+1} &= \frac{a}{2h_i^n \Delta\xi}\left(\overrightarrow{w}_{i+1/2-}^{n+1} - \overrightarrow{w}_{i-1/2-}^{n+1}\right), \\ (\mathcal{L}_{\overleftarrow{w}})_i^{n+1} &= -\frac{a}{2h_i^n \Delta\xi}\left(\overleftarrow{w}_{i+1/2+}^{n+1} - \overleftarrow{w}_{i-1/2+}^{n+1}\right) \end{aligned} \tag{3.52}$$

and we use the second order reconstruction introduced in (3.27).

The equations in (3.51) define two systems with four diagonals each: the ones corresponding to $\overrightarrow{w}_{i-2}$, $\overrightarrow{w}_{i-1}$, $\overrightarrow{w}_i$, $\overrightarrow{w}_{i+1}$ in the case of the system for $\overrightarrow{w}^{n+1}$ and the ones corresponding to $\overleftarrow{w}_{i-1}$, $\overleftarrow{w}_i$, $\overleftarrow{w}_{i+1}$, $\overleftarrow{w}_{i+2}$ in the case of the system for $\overleftarrow{w}^{n+1}$.

Theorem 3. *Given a water at rest stationary solution of system (1.1) with a continuous bottom topography z, the scheme defined by (3.46) and (3.51) is exactly well-balanced.*

Proof. The proof is analogous to the previous one. □

Implicit-explicit scheme Finally, the implicit-explicit second order scheme is obtained by treating the function $\mathcal{L}$ implicitly and the function $\mathcal{G}$ explicitly. To do so, we consider the SSP2(2,2,2) IMEX scheme defined by the Butcher tableau (see Pareschi and Russo [27]):

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}, \qquad \begin{array}{c|cc} \gamma & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}, \tag{3.53}$$

where $\gamma = 1 - \frac{\sqrt{2}}{2}$.

Similarly to the implicit second order case, the systems that we have to solved for this scheme have four diagonals each. Note that the use of the IMEX scheme avoids the costly nonlinear system inversion.

A similar theorem as Theorem 3 is still valid in this case.

4. The projection step

After solving the Lagrangian step, we shall project back the result onto Eulerian coordinates. This will be done explicitly. For doing the projection of $L\overline{\mathbf{U}}(\xi, t)$ onto the Eulerian cells $(x_{i-1/2}, x_{i+1/2})$, we need to compute

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x,t)dx.$$

Given $t \geq 0$ we define $\hat{\xi}_{i+1/2}(t)$ such that

$$x(\hat{\xi}_{i+1/2}(t), t) = x_{i+1/2}.$$

For any time $T \geq 0$, let us define $\hat{\xi}_{i+1/2}(T)$ as the origin of the curve $x(\hat{\xi}_{i+1/2}, t)$ which at time $t = T$ passes through $x_{i+1/2}$ (see Fig. 1).

Using this notation, we may write

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{x(\hat{\xi}_{i-1/2}(t),t)}^{x(\hat{\xi}_{i+1/2}(t),t)} \mathbf{U}(x,t)dx = \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}(t)}^{\hat{\xi}_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi,$$

and we can split the integral as follows

$$\mathbf{U}_i(t) = \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}(t)}^{\xi_{i-1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi + \frac{1}{\Delta x}\int_{\xi_{i-1/2}(t)}^{\xi_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi + \frac{1}{\Delta x}\int_{\xi_{i+1/2}(t)}^{\hat{\xi}_{i+1/2}(t)} L(\xi,t)\overline{\mathbf{U}}(\xi,t)d\xi.$$

Remark that the central integral corresponds to $(L\overline{\mathbf{U}})_i(t)$, which is already known from the previous Lagrangian step. Therefore,

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} + \frac{1}{\Delta x}\int_{\hat{\xi}_{i-1/2}}^{\xi_{i-1/2}} L(\xi, t^{n+1})\overline{\mathbf{U}}(\xi, t^{n+1})d\xi + \frac{1}{\Delta x}\int_{\xi_{i+1/2}}^{\hat{\xi}_{i+1/2}} L(\xi, t^{n+1})\overline{\mathbf{U}}(\xi, t^{n+1})d\xi. \tag{4.1}$$

The evaluation of the integrals in previous expressions is now required. To do so, first and second order approaches will be presented.

4.1. First order projection scheme

The previous integrals can be approximated in the following way:

$$\frac{1}{\Delta x}\int_{\xi_{i+1/2}}^{\hat{\xi}_{i+1/2}} L(\xi, t^{n+1})\overline{\mathbf{U}}(\xi, t^{n+1})d\xi = \frac{\hat{\xi}_{i+1/2} - \xi_{i+1/2}}{\Delta x}(L\overline{\mathbf{U}})_{i+1/2}^{n+1},$$

where

$$(L\overline{\mathbf{U}})_{i+1/2}^{n+1} = \begin{cases} (L\overline{\mathbf{U}})_i^{n+1} \text{ for } \xi_{i+1/2} > \hat{\xi}_{i+1/2}, \\ (L\overline{\mathbf{U}})_{i+1}^{n+1} \text{ for } \xi_{i+1/2} \le \hat{\xi}_{i+1/2}. \end{cases}$$

Moreover, using the approximation

$$\hat{\xi}_{i+1/2} = x_{i+1/2} - \Delta t u_{i+1/2}^*,$$

then from (4.1) we get

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^*(L\overline{\mathbf{U}})_{i+1/2}^{n+1} - u_{i-1/2}^*(L\overline{\mathbf{U}})_{i-1/2}^{n+1}\right).$$

4.2. Second order projection scheme

For the second order case, the integrals (4.1) need to be computed with second order accuracy. To do so, let us consider a piecewise linear reconstruction of averages of $(L\overline{\mathbf{U}})_i^{n+1}$ and, taking into account that the velocities $u_{i+1/2}^{*,n+1}$ that are continuously defined at the intercells, we use a continuous piecewise linear interpolation of at the intercells.

It can be seen that we can write (4.1) again as

$$\mathbf{U}_i^{n+1} = (L\overline{\mathbf{U}})_i^{n+1} - \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^*(L\overline{\mathbf{U}})_{i+1/2}^{n+1} - u_{i-1/2}^*(L\overline{\mathbf{U}})_{i-1/2}^{n+1}\right),$$

where

$$(L\overline{\mathbf{U}})_{i+1/2}^{n+1} = \begin{cases} (L\overline{\mathbf{U}})_{i+1/2-}^{n+1} \text{ for } u_{i+1/2}^* > 0, \\ (L\overline{\mathbf{U}})_{i+1/2+}^{n+1} \text{ for } u_{i+1/2}^* \le 0, \end{cases}$$

and

$$\left(L\overline{\mathbf{U}}\right)_{i+1/2-}^{n+1} = \frac{1}{L_i^{n+1}}\left(\left(L\overline{\mathbf{U}}\right)_i^{n+1} + \frac{1}{2}\left(\delta L\overline{\mathbf{U}}\right)_i^{n+1}\left(\Delta x - \frac{\Delta t}{L_i^{n+1}}u_{i+1/2}^{*,n+1}\right)\right),$$

$$\left(L\overline{\mathbf{U}}\right)_{i+1/2+}^{n+1} = \frac{1}{L_{i+1}^{n+1}}\left(\left(L\overline{\mathbf{U}}\right)_{i+1}^{n+1} + \frac{1}{2}\left(\delta L\overline{\mathbf{U}}\right)_{i+1}^{n+1}\left(-\Delta x - \frac{\Delta t}{L_{i+1}^{n+1}}u_{i+1/2}^{*,n+1}\right)\right).$$

In the previous expressions, $\left(\delta L\overline{\mathbf{U}}\right)_i^{n+1}$ and $\left(\delta L\overline{\mathbf{U}}\right)_{i+1}^{n+1}$ are approximations of the derivatives of $L\overline{\mathbf{U}}$ at time t^{n+1} at x_i and x_{i+1}, respectively, that are computing by means of a limiter. $(L\overline{\mathbf{U}})_{i-1/2}^{n+1}$ is defined in a similar way.

The CFL condition associated with the transport step reads

$$\Delta t \max_j \left\{(u_{j-1/2}^*)^+ - (u_{j+1/2}^*)^-\right\} \le \Delta x. \tag{4.2}$$

Let us remark that this stability restriction always remains as this step that, contrary to the Lagrangian one, is performed explicitly.

Observe that this projection step does not destroy the well-balanced character of the scheme as we consider only steady states corresponding to water at rest, where $u = 0$, and we have shown that $L_i = 1$, what makes the projection step trivial in that particular case. A more interesting situation occurs when moving equilibria are considered. In that case, the procedure described in Section 3 could be extended to moving equilibria following [25], but the projection step must be modified in order to preserve those steady states. This will be considered in a future work.

Table 1
L^1 errors between the numerical solution at initial and final time $t = 5$ for water at rest initial condition.

Order 1			Order 2		
EXP	IMP-IMEX	IMP-NL	EXP	IMP-IMEX	IMP-NL
1.95e−14	8.57e−14	5.36e−14	1.67e−13	5.56e−14	6.66e−14

5. Numerical results

We now intend to test and compare the different numerical schemes introduced in this work. In what follows, we shall use the following notation for the used schemes. First and second order explicit schemes, which are described in Appendix A, are denoted by EXP O1 and EXP O2 respectively. The first order implicit scheme given by (3.34)–(3.41) is denoted by IMP-NL O1 while its second order extension (3.46)–(3.51) is denoted by IMP-NL O2. Finally the first order (3.43)–(3.45) implicit-explicit scheme and its second order extension by means of (3.53) are represented by IMP-IMEX O1 and IMP-IMEX O2 respectively. In order to make a better assessment of the performance of our second order schemes, in some of the tests we will also include the results obtained with a DIRK (Diagonal Implicit Runge–Kutta) scheme [27] that we will represent by IMP-DIRK O2.

In what follows, CFL condition refers to the restriction needed to satisfy for the stability of the explicit schemes, that is, Δt is the minimum that grants the stability conditions given by (3.14) and (4.2). Note that both the fully implicit and the IMEX schemes do not need the restriction (3.14) and they are only limited by (4.2). This means that, in situations where velocity is small compared to the sound speed, the stability condition for the Lagrangian step (3.14) is very limiting when compared to the projection step stability condition (4.2), which justifies the use of the implicit approach. Therefore, we will see in the following test cases that CFL greater than 1 (when compared to the stability criterion for the explicit scheme) may be chosen for the implicit schemes.

5.1. Well-balanced property test

This first test aims at assessing the well balanced property of the schemes. Let us propose the water at rest solution given by:

$$u = 0, \;\; h + z = C, \;\; \text{with } C \in \mathbb{R}$$

where $C = 0$ and the bottom topography is a Gaussian bump given by

$$z(x) = -1 + \frac{1}{2} e^{-x^2}, \quad x \in [-5, 5]. \tag{5.1}$$

The interval $[-5, 5]$ is discretized using 200 cells and final time is set to $t = 5$. In Table 1 we show the L^1 errors obtained at final time using CFL 0.5 for the explicit schemes and CFL 2 for the implicit ones. As expected, all the well-balanced schemes are able to preserve the water at rest steady state.

5.2. Computational time vs. error

The objective of this paragraph is to show the better performance of IMEX schemes against the fully implicit approach. To do so, we study the computational efficiency of these schemes.

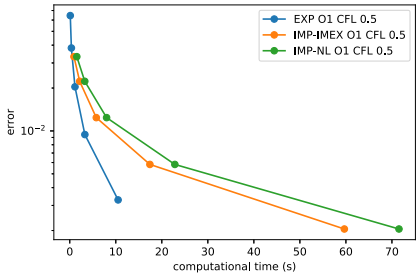
Inspired by Bermudez and Vazquez [2], we have considered a test case consisting in a channel of length $L = 14,000$ with a bottom topography defined by

$$z(x) = -\left(50.5 - 40\frac{L-x}{L} + 10 \sin\left(\pi\left(4\frac{L-x}{L} - \frac{1}{2}\right)\right)\right).$$

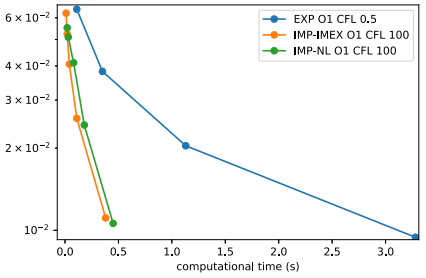
We then simulate a tidal wave of $0,5$ m amplitude by imposing the following initial and boundary conditions:

$$\begin{aligned} &h(x,0) = -z(x) + 1, \\ &q(x,0) = 0, \\ &h(L,t) = -z(L) + \frac{1}{2} + \frac{1}{2}\sin\left(\pi\left(\frac{4t}{86,400} + \frac{1}{2}\right)\right), \\ &q(0,t) = 0. \end{aligned}$$

Let us recall that the computational advantage of the numerical approaches presented in this work make sense especially in the case of low Froude number. Indeed, in such situations the restriction of the usual CFL condition is mainly driven by the acoustic waves, so that implicit schemes allow to avoid it. This is indeed the case of this tidal wave test.

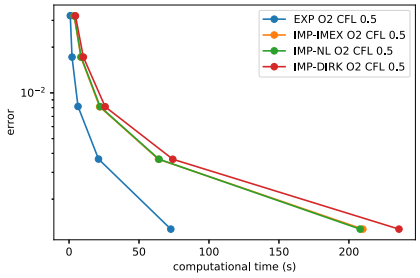


(a) Using a CFL value of 0.5 for the explicit and implicit schemes

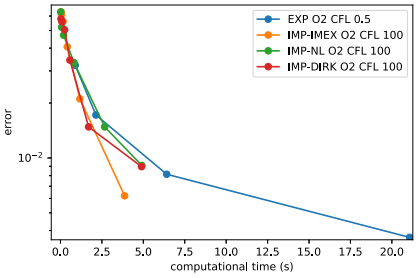


(b) Using a CFL value of 100 for the implicit schemes and 0.5 for the explicit scheme

Fig. 2. Computational time vs. error for an increasing number of cells using first order schemes.



(a) Using a CFL value of 0.5 for the explicit and implicit schemes



(b) Using a CFL value of 100 for the implicit schemes and 0.5 for the explicit scheme

Fig. 3. Computational time vs. error for an increasing number of cells using second order schemes.

In Figs. 2 and 3 we show the computational time needed to perform the simulation and the error for an increasing number of cells for the different type of schemes. The error has been obtained by computing a reference solution for a very fine mesh.

In all the numerical tests, a CFL value of 0.5 has been used in the case of explicit schemes. As far as the implicit schemes are concerned, we consider two different approaches: first, the simulations are performed using the same CFL restriction 0.5 as in the explicit case (left-hand side pictures); second a CFL corresponding to 100 with respect to the usual restriction is used (right-hand side pictures).

We can observe that for the low CFL value case, the explicit scheme is more efficient than the implicit ones, as expected. It is in the case when large CFL numbers are used that the implicit schemes outperform the explicit one. Indeed we see an important reduction of computational time needed for the implicit schemes when compared to the explicit one in order to obtain a similar error. For example, in the first order case, we see a 95% reduction in computational time for the IMP-IMEX scheme compared to the EXP one for comparable errors. Similarly, in the second order case, the reduction found is approximately 60%. This improvement in the efficiency is expected to be even bigger for 2D problems.

Moreover, when comparing IMP-IMEX and IMP-NL, the former seems to be more efficient than the latter. This is due to the iterations of the fixed point scheme that are needed. In Tables 2 and 3 we show the maximum number of iterations needed to solve the nonlinear systems during the computation and the global number of fixed point iterations required for the whole simulation. This is done for CFL equal to 0.5 and 100, respectively, for the IMP-NL O2 and the IMP-DIRK O2 schemes. We can also remark that the computational time required by the DIRK scheme is larger than in the IMP-NL O2 case when the CFL is set to 0.5 but it is a bit shorter when the CFL is 100. This can also be explained by checking the total number of iterations in Tables 2 and 3. However, the IMP-IMEX O2 scheme is faster than the IMP-NL and the IMP-DIRK schemes, since only two linear schemes have to be solved.

Table 2
Maximum number of fixed point iterations to solve the nonlinear systems and total number of iterations performed for different number of cells for the second order nonlinear scheme and the DIRK scheme using a CFL equal to 0.5.

CFL 0.5	IMP-NL O2		IMP-DIRK O2	
No. of cells	Max Iter	Total Iter	Max Iter	Total Iter
25	3	496	3	717
50	3	981	2	1342
100	3	1897	2	1682
200	3	3681	2	5366
400	3	6942	2	10736

Table 3
Maximum number of fixed point iterations to solve the nonlinear systems and total number of iterations performed for different number of cells for the second order nonlinear scheme and the DIRK scheme using a CFL equal to 100.

CFL 100	IMP-NL O2		IMP-DIRK O2	
No. of cells	Max Iter	Total Iter	Max Iter	Total Iter
25	2	2	1	1
50	7	9	6	7
100	7	21	7	18
200	7	39	6	34
400	7	82	6	73

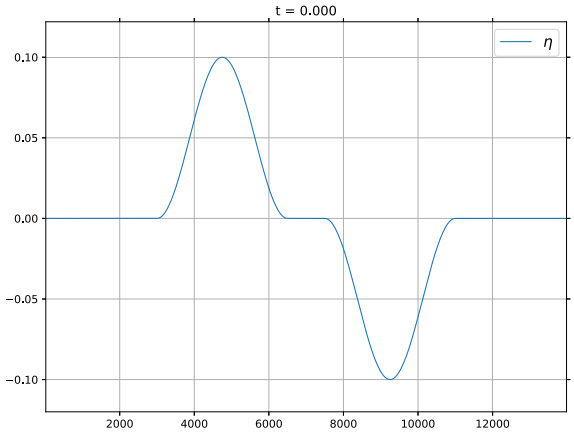


Fig. 4. Free surface corresponding to the initial condition for the order test case.

5.3. Order test

Let us now check the order of the schemes. In order to do so, we consider as initial condition a small perturbation flowing over a Gaussian bump in a domain with length $L = 14,000$ m. More explicitly, the bottom topography is given by

$$z(x) = -\left(50 - \exp\left(-\frac{(x-7000)^2}{1,000,000}\right)\right),$$

and the initial condition writes as $q(x,0) = 0$ and

$$h(x,0) = \begin{cases} 0.05\left(1+\cos\left(\frac{2\pi\,(x-4750)}{3500}\right)\right) & \text{if} \quad 3000 < x < 6500 \\ 0.05\left(-\left(1+\cos\left(\frac{2\pi\,(x-9250)}{3500}\right)\right)\right) & \text{if} \quad 7500 < x < 11,000 \\ 0 & \text{otherwise} \end{cases}$$

Free surface corresponding to this initial condition is shown in Fig. 4. The errors in L^1 obtained for the different schemes are then shown in Tables 4–6, where we see that the expected accuracy is obtained.

Table 4
Dimensionless errors in L^1 norm and convergence rates for the explicit LP schemes with CFL value 0.5.

No. of cells	EXP - Order 1				EXP - Order 2			
	h		q		h		q	
	Error	Order	Error	Order	Error	Order	Error	Order
25	3.49e−1	0.00	4.50e0	0.00	2.17e−1	0.00	1.58e0	0.00
50	1.88e−1	0.89	2.69e0	0.74	6.03e−2	1.85	6.89e−1	1.20
100	9.39e−2	1.00	1.35e0	0.99	1.68e−2	1.84	2.01e−1	1.77
200	4.21e−2	1.16	6.07e−1	1.16	4.17e−3	2.01	5.42e−2	1.89
400	1.44e−2	1.54	2.07e−2	1.54	8.50e−4	2.30	1.17e−2	2.21

Table 5
Dimensionless errors in L^1 norm and convergence rates for the implicit IMEX LP schemes with CFL value 3.

No. of cells	IMP-IMEX - Order 1				IMP-IMEX - Order 2			
	h		q		h		q	
	Error	Order	Error	Order	Error	Order	Error	Order
25	7.03e−1	0.00	1.37e1	0.00	6.53e−1	0.00	3.60e0	0.00
50	5.25e−1	0.42	9.39e0	0.55	1.66e−1	1.97	3.55e0	0.02
100	3.78e−1	0.47	5.85e0	0.68	4.78e−2	1.80	1.16e0	1.61
200	2.18e−1	0.79	3.01e0	0.96	1.30e−2	1.87	3.21e−1	1.86
400	8.89e−2	1.30	1.16e0	1.37	3.22e−3	2.02	7.21e−2	2.16

Table 6
Dimensionless errors in L^1 norm and convergence rates for the implicit nonlinear LP schemes with CFL value 3.

No. of cells	IMP-NL - Order 1				IMP-NL - Order 2			
	h		q		h		q	
	Error	Order	Error	Order	Error	Order	Error	Order
25	7.03e−1	0.00	1.37e1	0.00	4.57e−1	0.00	5.61e0	0.00
50	5.25e−1	0.42	9.39e0	0.55	1.60e−1	1.51	3.89e0	0.52
100	3.78e−1	0.47	5.85e0	0.68	5.46e−2	1.55	1.19e0	1.30
200	2.18e−1	0.79	3.01e0	0.96	1.68e−2	1.70	4.53e−1	1.80
400	8.89e−2	1.30	1.16e0	1.37	4.14e−3	2.02	1.06e−1	2.09

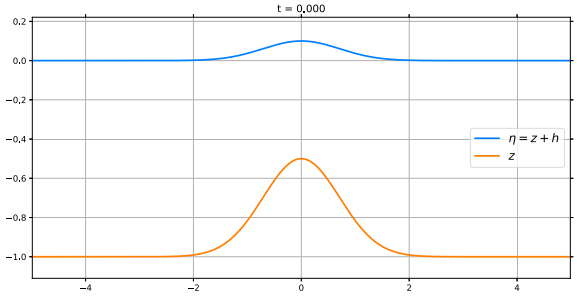


Fig. 5. Perturbation of water at rest initial condition.

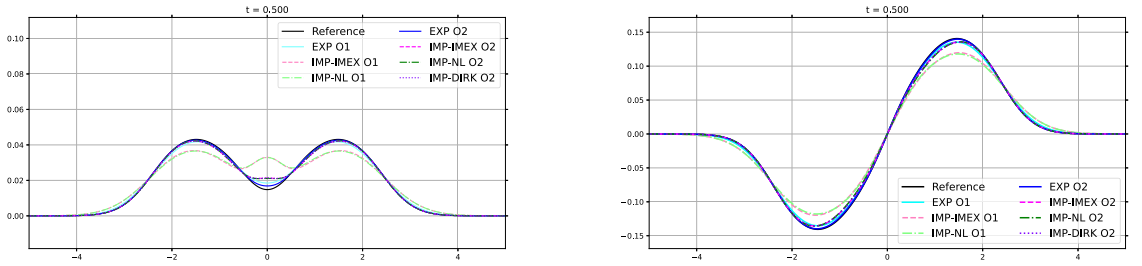
5.4. Perturbation of water at rest

Let us consider $z(x)$ given by (5.1) and the following initial condition, which is a perturbation of the water at rest:

$$h(x) = -z(x) + 0.1e^{-x^2}, \quad u(x) = 0.$$

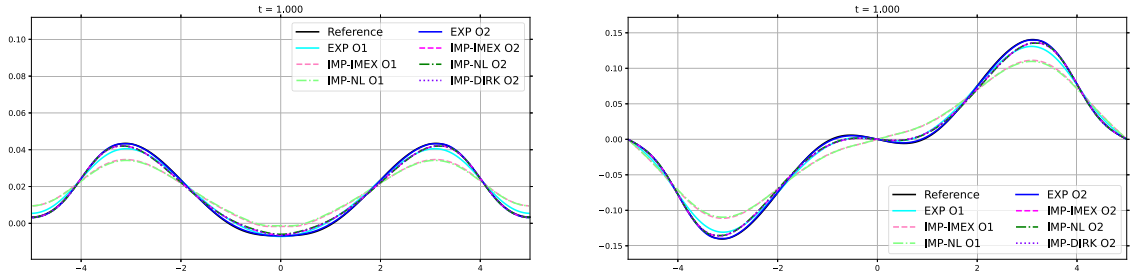
This initial condition is shown in Fig. 5.

In Figs. 6–9 we can check the solutions obtained with the different schemes at time $t = 0.5$ and $t = 1$ with 200 cells in the interval $[-5, 5]$ for the free surface $\eta = h + z$ and the discharge q. We include a reference solution that has been computed with the first order explicit scheme using 1600 cells.



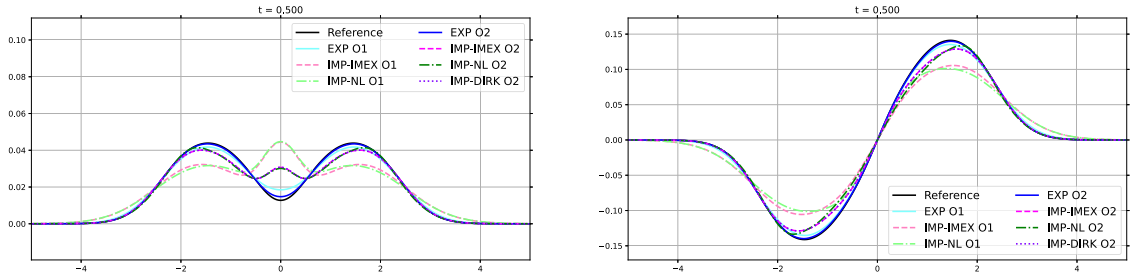
(a) $\eta = z + h$ (b) q

Fig. 6. Solution for η and q at $t = 0.5$ with 200 cells. Explicit: CFL=0.5, Implicit: CFL=2.



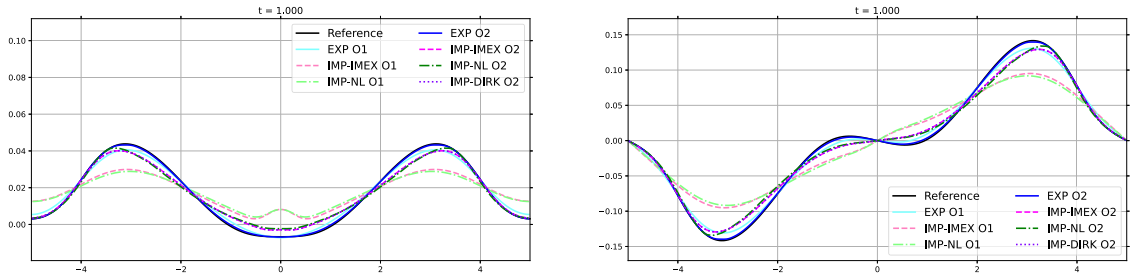
(a) $\eta = z + h$ (b) q

Fig. 7. Solution for η and q at $t = 1$ with 200 cells. Explicit: CFL = 0.5, Implicit: CFL = 2.



(a) $\eta = z + h$ (b) q

Fig. 8. Solution for η and q at $t = 0.5$ with 200 cells. Explicit: CFL = 0.5, Implicit: CFL = 5.



(a) $\eta = z + h$ (b) q

Fig. 9. Solution for η and q at $t = 1$ with 200 cells. Explicit: CFL = 0.5, Implicit: CFL = 5.

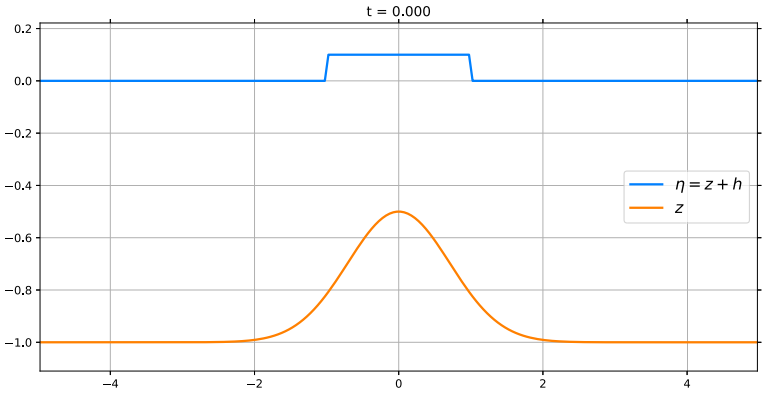
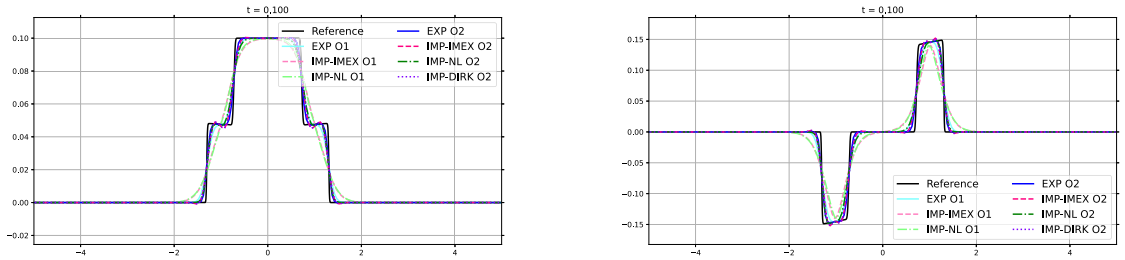


Fig. 10. water at rest solution with a discontinuous perturbation on the surface.



(a) $\eta = z + h$ (b) q

Fig. 11. Solution at $t = 0.1$ with 200 cells for a discontinuous perturbation of a water at rest. Explicit: CFL = 0.5, Implicit: CFL = 2.

In general we see a good agreement of the numerical solutions when compared to the reference solution. The first order implicit schemes are the most diffusive, which is expected. Also, the second order implicit schemes present more diffusion when the CFL is increased. Nevertheless, the greater CFL allowed by implicit schemes make them more efficient.

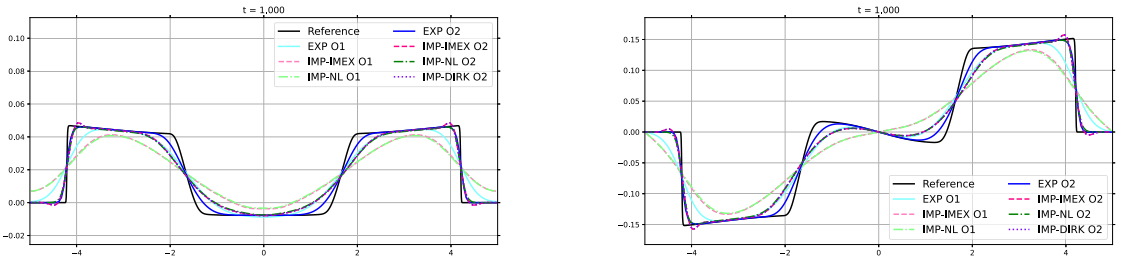
5.5. Perturbed water at rest with shock waves

We now intend to study the behavior of the schemes, specially for second order, in the presence of shocks. In such situation, the limiter will play an important role. We consider $z(x)$ given by (5.1) in $[-5, 5]$ and we set the initial condition:

$$h(x) = \begin{cases} -z(x) & if \quad |x| \geq 1 \\ -z(x) + 0.1 & if \quad |x| < 1 \end{cases}, \quad u(x) = 0.$$

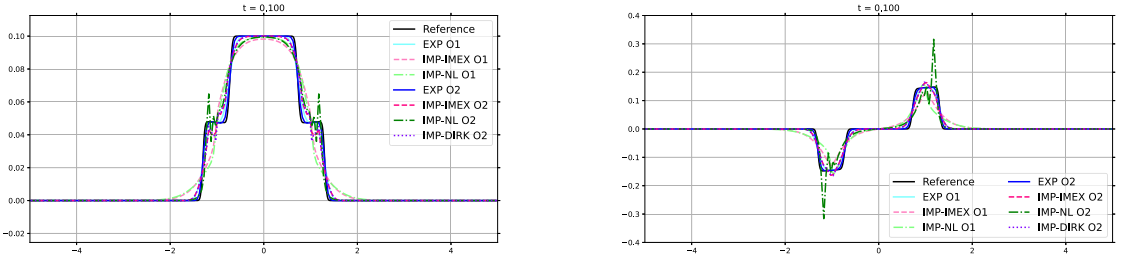
This initial condition, shown in Fig. 10, presents two discontinuities at the interface that generate two shock waves travelling in different directions. We consider a mesh consisting on 200 cells and we perform the simulation up to time $t = 1$. For the explicit schemes we have set the CFL value to be 0.5, and for the implicit ones it is set to be 2.

Figs. 11 –14 show the numerical solution obtained at times $t = 0.1$ and $t = 1$ seting the CFL value to be 2 and 5 for the implicit schemes. As in the previous test, the reference solution has been computed by using the first order explicit scheme with 1600 cells. We see that the schemes are able to correctly handle the initial condition, although very small spurious oscillations are seen for the second order schemes at the advancing front of the shock waves. These spurious oscillations are more pronounced for the second order IMP-NL than for the IMP-IMEX. Nevertheless, in the CFL 2 case, they are mostly suppressed thanks to the presence of the slope limiter. These oscillations become bigger for larger CFL values. This drawback has been pointed out at least in Spijker [28], Gottlieb et al. [29]. Some recent strategies that could be considered in order to reduce such oscillations can be found in Puppo et al. [30], Gottlieb et al. [31], Michel-Dansac and Thomann [32].



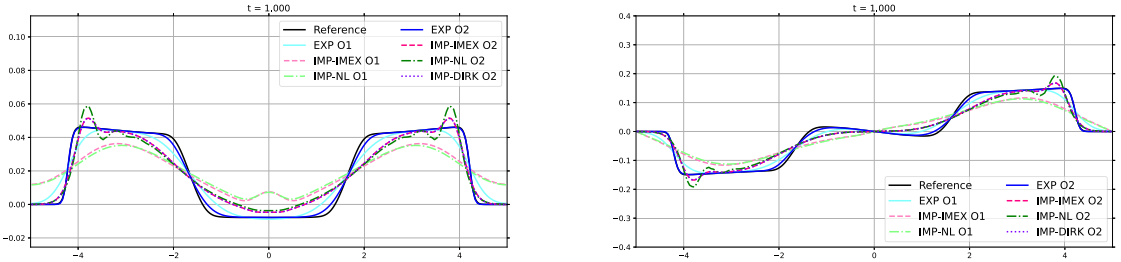
(a) $\eta = z + h$ (b) q

Fig. 12. Solution at $t = 1$ with 200 cells for a discontinuous perturbation of a water at rest. Explicit: CFL = 0.5, Implicit: CFL = 2.



(a) $\eta = z + h$ (b) q

Fig. 13. Solution at $t = 0.1$ with 200 cells for a discontinuous perturbation of a water at rest. Explicit: CFL = 0.5, Implicit: CFL = 5.



(a) $\eta = z + h$ (b) q

Fig. 14. Solution at $t = 1$ with 200 cells for a discontinuous perturbation of a water at rest. Explicit: CFL = 0.5, Implicit: CFL = 5.

5.6. Generation of subcritical steady state

Let us consider again the bottom topography (5.1) and a water at rest as initial condition in the interval $[-5, 5]$:

$$\eta = h + z = 0,\ u = 0.$$

At the left boundary, we impose a discharge given by $q(x = -5, t) = 0.5$ and at the right boundary we fix the water height to $h(x = 5, t) = 1$, following the test proposed in Goutal and Maurel [33]. This boundary conditions are set using a ghost-cell technique. The results obtained at time $t = 2$, $t = 50$ and $t = 100$ for the different methods with 200 cells, with CFL value 0.5 for the explicit schemes and CFL value 2 and for the implicit ones, can be seen in Figs. 15–20. Due to the imposed discharge on the left boundary, a shock wave enters the domain and travels over the bump. The solution evolves until a subcritical steady state is reached. We remark that the front of the advancing shock is too diffusive for the first order schemes. Moreover, the location of the depression on the surface at times $t = 50$ and $t = 100$ is slightly misplaced in the case of the implicit first order schemes. The left water level is not the same as well. This is due to the higher diffusion of the schemes, which is overcome when using a second order scheme.

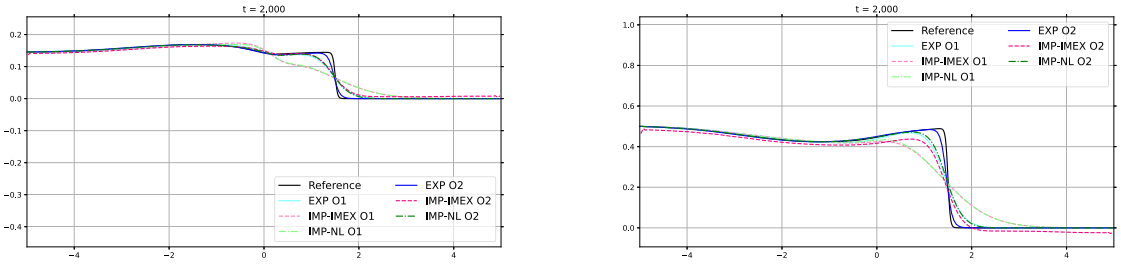


Fig. 15. Generation of subcritical steady state a time $t = 2$. Explicit: CFL = 0.5, Implicit: CFL = 2.

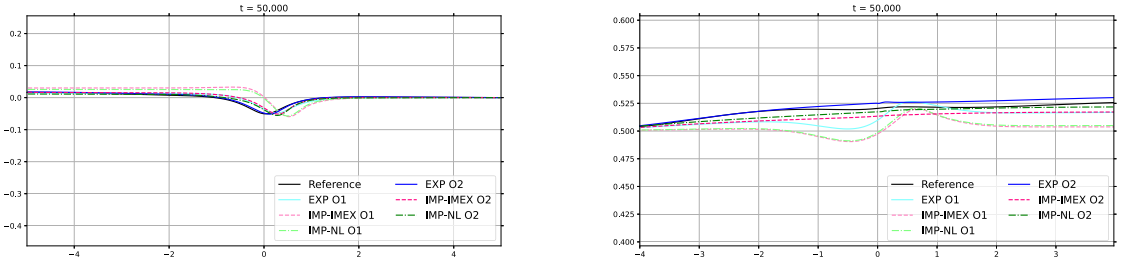


Fig. 16. Generation of subcritical steady state a time $t = 50$, Explicit: CFL = 0.5, Implicit: CFL = 2.

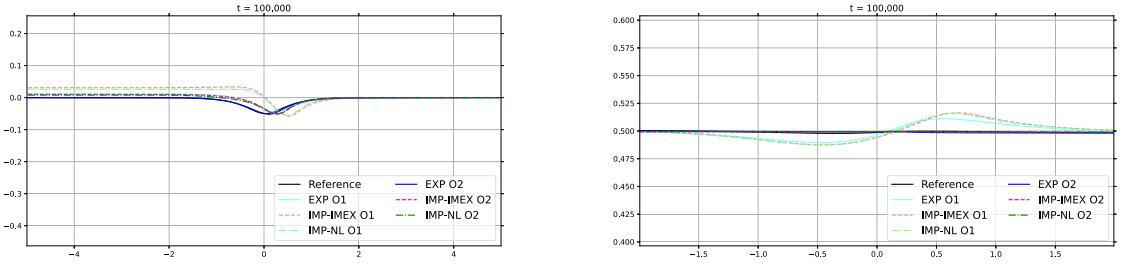


Fig. 17. Generation of subcritical steady state a time $t = 100$. Explicit: CFL = 0.5, Implicit: CFL = 2.

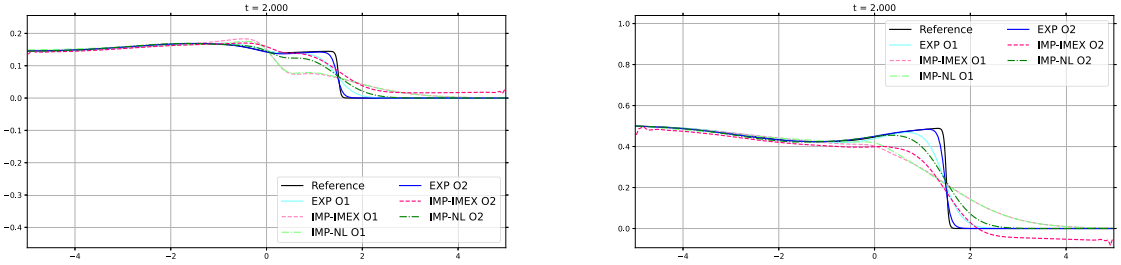


Fig. 18. Generation of subcritical steady state a time $t = 2$. Explicit: CFL = 0.5, Implicit: CFL = 5.

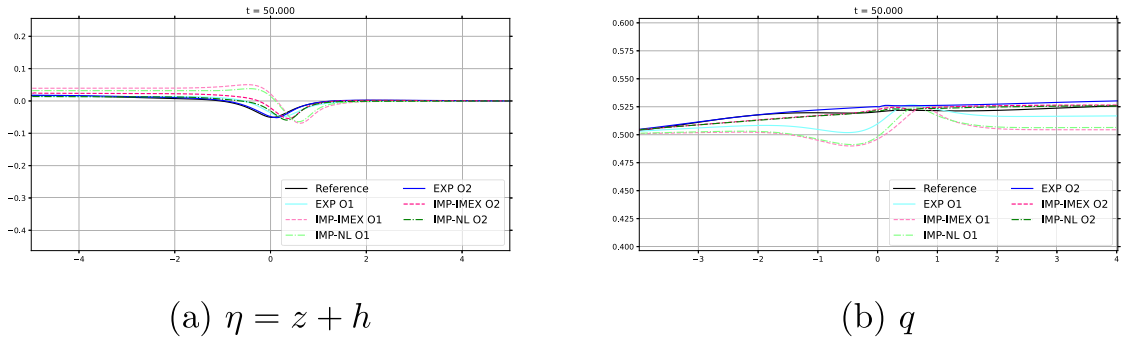


Fig. 19. Generation of subcritical steady state a time $t = 50$, Explicit: CFL $= 0.5$, Implicit: CFL $= 5$.

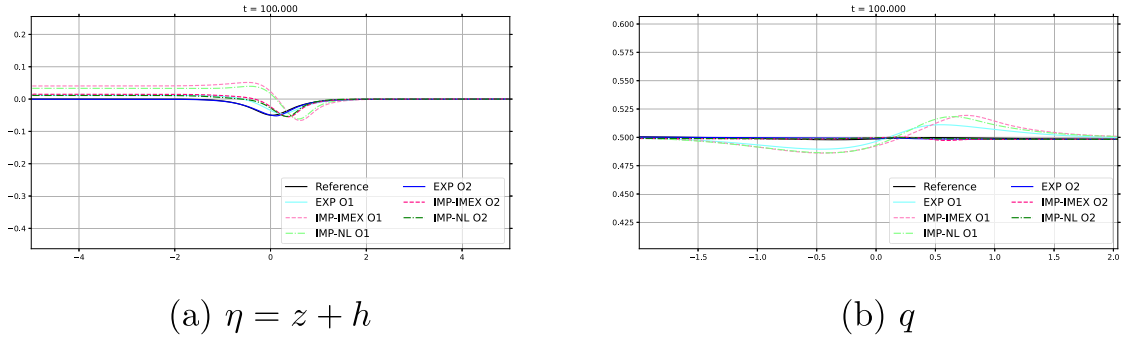


Fig. 20. Generation of subcritical steady state a time $t = 100$. Explicit: CFL $= 0.5$, Implicit: CFL $= 5$.

6. Conclusions

We have proposed first and second order Lagrange-Projection schemes for SWE where the Lagrangian step is performed following an implicit or implicit-explict approach. The latter has the advantage of being more efficient, as only linear systems need to be solved, while the former needs fixed point type iterations. All of them are exactly well-balanced for water at rest steady states. As expected, first order schemes are diffusive, which justifies the need of their second order counterpart. These schemes have proven to be very efficient to deal with subsonic or near low-Froude number flows, where the usual CFL time step limitation of Godunov-type schemes is indeed driven by the acoustic waves and can thus be very restrictive. The implicit and implicit-explicit Lagrange-Projection schemes avoid such restriction and allow to take large time-step with a CFL restriction based on the (slow) transport waves and not on the (fast) acoustic waves.

Concerning future work, we plan to extend the numerical schemes presented here to more general steady states, that is those corresponding to smooth moving equilibria steady states. Another point of interest for future work will be to adapt the some of the strategies developed in Puppo et al. [30], Gottlieb et al. [31], Michel-Dansac and Thomann [32] to reduce the oscillations in presence of discontinuities for big CFL numbers.

Data availability

Data will be made available on request.

Acknowledgments

This work is partially supported by projects RTI2018-096064-B-C21 and RTI2018-096064-B-C22 funded by MCIN/AEI/10.13039/501100011033 and “ERDF A way of making Europe”, projects P18-RT-3163 of Junta de Andalucía and UMA18-FEDERJA-161 of Junta de Andalucía-FEDER-University of Málaga. C. Caballero-Cárdenas is supported by the grant FPI2019/087773 funded by MCIN/AEI/10.13039/501100011033 and “ESF Investing in your future”. Funding for open access charge: Universidad de Málaga/CBUA.

Appendix A. Explicit Lagrangian schemes

In this appendix we will describe, for the sake of completeness, the proposed first and second order explicit Lagrangian schemes.

A1. First order explicit Lagrangian scheme

The first order explicit scheme can be written similarly to how it has been done for the implicit schemes, with the difference that in this case functions $\mathcal{L}$ and $\mathcal{G}$ are evaluated at time t^n.

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \Delta t\left(\mathcal{L}_i^n + \mathcal{G}_i^n\right),$$

where

$$\mathcal{L}_i^n = \frac{1}{\Delta\xi}\left(\pi_{i+1/2}^{*,n} - \pi_{i-1/2}^{*,n}\right),$$

and

$$\mathcal{G}_i^n = -\frac{1}{\Delta\xi}\left(\pi_i^{e,n}(x_{i+1/2}) - \pi_i^{e,n}(x_{i-1/2})\right).$$

Moreover,

$$L_i^n = 1 + \frac{\Delta t}{\Delta\xi}\left(u_{i+1/2}^{*,n} - u_{i-1/2}^{*,n}\right).$$

The needed approximations of $\pi_{i+1/2}^{*,n}$ and $u_{i+1/2}^{*,n}$ are obtained as

$$\pi_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o1}(\xi_{i+1/2},t^n) + P_{i+1,\overleftarrow{w}}^{o1}(\xi_{i+1/2},t^n)}{2} = \frac{\overrightarrow{w}_{i+1/2-}^n + \overleftarrow{w}_{i+1/2+}^n}{2}$$

and

$$u_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o1}(\xi_{i+1/2},t^n) - P_{i+1,\overleftarrow{w}}^{o1}(\xi_{i+1/2},t^n)}{2a} = \frac{\overrightarrow{w}_{i+1/2-}^n - \overleftarrow{w}_{i+1/2+}^n}{2a}$$

where

$$\overrightarrow{w}_{i+1/2-}^n = P_{i,\overrightarrow{w}}^{o1}(\xi_{i+1/2},t^n) = \pi_i^{e,n}(\xi_{i+1/2}) - \pi_i^{e,n}(\xi_i) + \overrightarrow{w}_i^n,$$
$$\overleftarrow{w}_{i+1/2+}^n = P_{i+1,\overleftarrow{w}}^{o1}(\xi_{i+1/2},t^n) = \pi_{i+1}^{e,n}(\xi_{i+1/2}) - \pi_{i+1}^{e,n}(\xi_i) + \overleftarrow{w}_{i+1}^n.$$

A2. Second order explicit Lagrangian scheme

For the second order explicit case we propose using a mid-point quadrature rule in time in order to obtain

$$(L\overline{hu})_i^{n+1} = (hu)_i^n - \Delta t\left(\mathcal{L}_i^{n+1/2} + \mathcal{G}_i^{n+1/2}\right),$$

where

$$\mathcal{L}_i^{n+1/2} = \frac{1}{\Delta\xi}\left(\pi_{i+1/2}^{*,n+1/2} - \pi_{i-1/2}^{*,n+1/2}\right),$$
$$\mathcal{G}_i^{n+1/2} = -\frac{1}{\Delta\xi}\left(\pi_i^e(x_{i+1/2}^{*,n+1/2}) + \pi_i^e(x_{i-1/2}^{*,n+1/2})\right) + g\left(h_i^n - L_i^{n+1/2}h^e(x_i^{*,n+1/2})\right)z'(x_i^{*,n+1/2})$$

and

$$L_i^{n+1/2} = 1 + \frac{\Delta t}{2\Delta\xi}(u_{i+1/2}^{*,n} - u_{i-1/2}^{*,n}).$$

The values $\pi_{i+1/2}^{*,n}$ and $u_{i+1/2}^{*,n}$ are approximated following the idea used for the implicit schemes:

$$\pi_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2},t^n) + P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2},t^n)}{2} = \frac{\overrightarrow{w}_{i+1/2-}^n + \overleftarrow{w}_{i+1/2+}^n}{2},$$

$$u_{i+1/2}^{*,n} = \frac{P_{i,\overrightarrow{w}}^{o2}(\xi_{i+1/2},t^n) - P_{i+1,\overleftarrow{w}}^{o2}(\xi_{i+1/2},t^n)}{2a} = \frac{\overrightarrow{w}_{i+1/2-}^n - \overleftarrow{w}_{i+1/2+}^n}{2a}$$

where for the reconstructions $\overrightarrow{w}_{i+1/2-}^n$, $\overleftarrow{w}_{i+1/2+}^n$ we use (3.27).

Once we have computed the previous values, we can obtain the approximations of the position at time $t^{n+1/2}$ of characteristic curves as

$$x_i^{*,n+1/2} = \xi_i + \frac{\Delta t}{4}\left(u_{i-1/2}^{*,n} + u_{i+1/2}^{*,n}\right),$$
$$x_{i\pm1/2}^{*,n+1/2} = \xi_{i\pm1/2} + \frac{\Delta t}{2} u_{i\pm1/2}^{*,n}.$$

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