CONVERGENCE OF THE LACUNARY ERGODIC CESÀRO AVERAGES

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Abstract. Let T be a positive linear operator with positive inverse. We consider in this paper the ergodic Cesàro-α averages $\mathcal{A}_{n,\alpha}f$, $0<\alpha\le 1$, and the ergodic Cesàro-α maximal operator associated to T. For Lebesgue spaces $L^p(\nu)$, it is known that the good range for the convergence of the Cesàro-α averages and the boundedness of the maximal operator is $1/\alpha<p<+\infty$. In this paper we study the convergence of $\mathcal{A}_{n_k,\alpha}f$, where $\{n_k\}$ is a lacunary sequence, and the boundedness of its associated ergodic maximal operator. We get positive results in the range $1\le p<\frac{1}{1-\alpha}$. We use transference arguments which leads to us to study in depth weighted inequalities of the lacunary Cesàro-α maximal operator in the setting of the integers and in the setting of the real line.

1. Introduction.

Let $(X,\mathcal{F},\nu)$ be a σ-finite measure space and let T be a linear operator acting on measurable functions. The ergodic Cesàro-α averages, $0<\alpha\le 1$, and the ergodic Cesàro-α maximal operator associated to T are defined by

$$\mathcal{A}_{n,\alpha}f=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}T^kf \quad \text{and} \quad \mathcal{M}_\alpha f=\sup_n \mathcal{A}_{n,\alpha}|f|,$$

respectively, where $A_n^\beta=\frac{(\beta+1)\cdots(\beta+n)}{n!}$ and $A_0^\beta=1$ for all $\beta>-1$. The Cesàro-β numbers, $\beta>-1$, have the following properties (see e.g. [19]):

(i) The numbers A_n^β are positive, increasing (as a function of n) for $\beta>0$ and decreasing for $-1<\beta<0$.
(ii) $\sum_{k=0}^{n}A_k^\beta=A_n^{\beta+1}$.
(iii) There exist positive constants C_1 and C_2, depending only on β, such that for all $n\ge 0$

$$C_1(n+1)^\beta\le A_n^\beta\le C_2(n+1)^\beta.$$

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R. Irmisch [10] proved that if T is a positive linear contraction on $L^p(\nu)$ and $p > 1/\alpha$ then the ergodic Cesàro-α maximal operator is bounded on $L^p(\nu)$,

$$||\mathcal{M}_\alpha f||_{L^p(\nu)} \leq \frac{p}{p-1}||f||_{L^p(\nu)},$$

and the averages $\mathcal{A}_{n,\alpha}f$ converge almost everywhere and in the $L^p(\nu)$-norm for all $f \in L^p(\nu)$. Notice that for $\alpha = 1$ Irmisch's result is the well-known Akcoglu's theorem [1].

If $\tau : X \to X$ is a measure preserving transformation then Irmisch's theorem can be applied to the positive operator (isometry) $Tf(x) = f(\tau x)$. Déniel [7] proved that the result does not hold in the limit case $p = \frac{1}{\alpha}$ and $\alpha < 1$. In fact, it was proved that if τ is ergodic and the measure space is finite and non atomic then there exists $f \in L^{\frac{1}{\alpha}}(\nu)$ such that the sequence of averages diverges a.e.. Broise, Déniel and Derriennic established [5] that if $\tau : X \to X$ is a measure preserving transformation then the limit of the averages $\mathcal{A}_{n,\alpha}f$ exists almost everywhere for any f in the Lorentz space $L_{1/\alpha,1}(\nu)$. Furthermore, $\mathcal{M}_\alpha$ is of restricted weak type $(1/\alpha, 1/\alpha)$, that is, there exists $C > 0$ such that

$$\nu(\{x : \mathcal{M}_\alpha f(x) > \lambda\}) \leq \frac{C}{\lambda^{1/\alpha}}||f||^{1/\alpha}_{L_{1/\alpha,1}(\nu)},$$

for all $\lambda > 0$ and all $f \in L_{1/\alpha,1}(\nu)$, where $||f||_{L_{1/\alpha,1}(\nu)}$ is the usual quasi-norm in the Lorentz space.

The above results were extended to a more general kind of operators in [13] and [3]. Essentially, these operators will be the setting in which we shall present the theorems of this paper. In order to introduce these operators let us consider a non singular measurable invertible transformation τ on X, that is, $\tau : X \to X$ is a map, $\tau^{-1}E \in \mathcal{F}$ if and only if $E \in \mathcal{F}$ and $\nu(\tau^{-1}E) = 0$ if and only if $\nu(E) = 0$. Let g be a positive measurable function and let T be the operator induced by g and τ,

$$Tf(x) = g(x)f(\tau x). \tag{1.1}$$

It is clear that $T^i f(x) = g_i(x)f(\tau^i x)$, where $g_0(x) = 1$ and the functions g_i satisfy $g_{i+j}(x) = g_i(x)g_j(\tau^i x)$.

The measures $\nu_i(E) = \nu(\tau^i E)$ have the same sets of measure zero as ν. If J_i is the Radon-Nikodym derivative of ν_i with respect to ν we have that $J_{i+j}(x) = J_i(x)J_j(\tau^i x)$ and if $H_i(x) = (g_i(x))^{-p}J_i(x)$ the following key property

$$\int_X |f(x)|^p \, d\nu(x) = \int_X |T^i f(x)|^p H_i(x) \, d\nu(x) \tag{1.2}$$

holds for all nonnegative functions and for all $f \in L^p(\nu)$. The following relevant result was proved in [14] in a slightly more general setting.

Theorem 1.1. [14] *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, τ a non singular measurable invertible transformation on X, g a positive measurable function and T the operator defined by (1.1). Let $1 < p < \infty$. The following are equivalent.*

(a) $\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} T^k \right\|_{L^p(\nu)} < \infty$.
(b) *for a.e. $x \in X$ the function $H_x \in A_p^+$ with the same A_p^+ constant, where $H_x : \mathbb{Z} \to \mathbb{R}$, $H_x(i) = H_i(x)$ and A_p^+ is the discrete Sawyer's condition (see Section 2).*
(c) *The ergodic maximal operator $\mathcal{M}_1 = \mathcal{M}$ is bounded on $L^p(\nu)$.*

We have also that the adjoint of T is the operator

$$T^*f(x) = \frac{J_{-1}(x)}{g_{-1}(x)} f(\tau^{-1}x). \tag{1.3}$$

For this kind of operators T, it was proved in [13] that the Cesàro averages behave well in the good range $p > 1/\alpha$ under the assumption that the averages of a modification of T are uniformly bounded in $L^{p\alpha}(\nu)$. We quote the result.

Theorem 1.2. [13] *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, $\frac{1}{\alpha} < p < \infty$, τ a non singular measurable invertible transformation on X, g a positive measurable function, T the operator defined by (1.1) and $T_\alpha f(x) = (g(x))^{1/\alpha} f(\tau x)$. Assume that*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} T_\alpha^k \right\|_{L^{p\alpha}(\nu)} < \infty \ . \tag{1.4}$$

Then there exists $C > 0$ such that

$$||\mathcal{M}_\alpha f||_{L^p(\nu)} \leq C||f||_{L^p(\nu)},$$

for all $f \in L^p(\nu)$. Further, the sequence of averages $\mathcal{A}_{n,\alpha} f$ converges almost everywhere and in the $L^p(\nu)$-norm for all $f \in L^p(\nu)$ as $n \to \infty$.

Remark 1.3. Theorem 1.2 also holds replacing assumption (1.4) by the following apparently weaker assumption: for almost every x the function $i \to H_i(x)$ on the integers satisfies the condition $A^+_{p,\alpha}$ (see the definition of the condition in §2).

In the limit case $p = 1/\alpha$, we have to assume $g = 1$ and the result is in the following theorem (notice that it follows from Theorem 1.4 in [3]).

Theorem 1.4. [3] *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, τ a non singular measurable invertible transformation on X and $Tf(x) = f(\tau x)$. Assume that*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} T^k \right\|_{L^1(\nu)} < \infty \ . \tag{1.5}$$

Then there exists $C > 0$ such that

$$\nu(\{x : \mathcal{M}_\alpha f(x) > \lambda\}) \leq \frac{C}{\lambda^{1/\alpha}} ||f||^{1/\alpha}_{L_{1/\alpha,1}(\nu)}$$

for all $\lambda > 0$ and all $f \in L_{1/\alpha,1}(\nu)$. Further, the sequence of averages $\mathcal{A}_{n,\alpha} f$ converges almost everywhere and in measure for all $f \in L_{1/\alpha,1}(\nu)$ as $n \to \infty$.

Remark 1.5. Notice that (1.5) holds if and only if for almost every $x \in X$, the functions $w_x(i) = J_i(x)$ on the integers satisfy that $w_x \in A_1^+$ with a constant independent of x.

Clearly, for $\alpha < 1$, the range $p \geq 1/\alpha$, is the good range for the convergence of the full sequence $\mathcal{A}_{n,\alpha}$. We can not obtain positive results out of this range unless we take subsequences $\mathfrak{n} = \{n_k\}$. With this idea we are going to consider ρ-lacunary sequences $\mathfrak{n}$ of positive integers, which generally have a better behavior (see for instance the corollary in [18, p.75] for the sequence $\{2^k\}$). In this paper we say that $\mathfrak{n} = \{n_k\}$ is a $\rho - lacunary$ sequence, $\rho > 1$, if

$$\rho \leq \frac{n_{k+1}}{n_k} \leq \rho^2 \tag{1.6}$$

for all $k \in \mathbb{N}$ and $n_1 = 1$. The typical example is $n_k = 2^{k-1}$. Observe that the following properties hold:

$$\rho^{j-i} \leq \frac{n_j}{n_i} \leq \rho^{2(j-i)}, \quad \text{for all} \quad j > i. \tag{1.7}$$

If we denote by β the smallest positive integer such that $1/\rho + (1/\rho)^\beta \leq 1$, we get from (1.7) that

$$n_i + n_j \leq n_{j+1} \quad \text{for all} \quad j \geq i + \beta - 1. \tag{1.8}$$

Given a $\rho-$*lacunary* sequence $\mathfrak{n} = \{n_k\}$, we consider the ergodic Cesàro-α averages $\mathcal{A}_{n_k,\alpha} f$ associated to this sequence and the lacunary ergodic maximal operator,

$$\mathcal{M}_{\alpha,\mathfrak{n}} f(x) = \sup_{k\in\mathbb{N}} \mathcal{A}_{n_k,\alpha}|f|(x). \tag{1.9}$$

The following results show sufficient conditions for the convergence of the subsequences $\mathcal{A}_{n_k,\alpha} f$ for $f \in L^p(\nu)$ with $1 \leq p < \frac{1}{1-\alpha}$. (Notice that $\frac{1}{\alpha} \leq \frac{1}{1-\alpha}$ if and only if $\alpha \geq 1/2$.) We state first the case $p = 1$ and then the case $1 < p < \frac{1}{1-\alpha}$.

Theorem 1.6. *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence, τ an invertible non-singular measurable transformation and $Tf(x) = f(\tau x)$. Let $\widetilde{T_\varepsilon} f(x) = (J_1(x))^{-\varepsilon} f(\tau x)$ for some $\varepsilon > 0$. If*

$$\sup_{k\in\mathbb{N}} \left\| \frac{1}{A_{n_k}^\alpha} \sum_{j=0}^{n_k} A_{n_k-j}^{\alpha-1} \widetilde{T_\varepsilon}^j \right\|_{L^1(\nu)} = M_1 < \infty, \tag{1.10}$$

then there exists $C > 0$ such that

$$\nu(\{x : \mathcal{M}_{\alpha,\mathfrak{n}} f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| \, d\nu \tag{1.11}$$

for all $\lambda > 0$ and all $f \in L^1(\nu)$. Furthermore, the limit $\lim_{k\to\infty} \mathcal{A}_{n_k,\alpha} f$ exists almost everywhere and in measure, for any f in $L^1(\nu)$.

Remark 1.7. Notice that ε can be very small; consequently the operator $\widetilde{T_\varepsilon}$ in Theorem 1.6 can be considered as a small perturbation of T.

Corollary 1.8. Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, τ an invertible non-singular measurable transformation and $Tf(x) = f(\tau x)$. If

$$\sup_{n\geq 0} \left\| \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k \right\|_{L^1(\nu)} < \infty, \tag{1.12}$$

then the conclusions in Theorem 1.6 hold for all ρ-lacunary sequences $\mathfrak{n}$. In particular, the conclusions hold if τ is a measure preserving transformation.

Theorem 1.9. *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence, $1 < p < \frac{1}{1-\alpha}$, τ an invertible non-singular measurable transformation, g a positive measurable function and $Tf(x) = g(x) f(\tau x)$. Let $T_\alpha^* f(x) = \left(\frac{J_{-1}(x)}{g_{-1}(x)}\right)^{1/\alpha} f(\tau^{-1}x)$. If*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} (T_\alpha^*)^k \right\|_{L^{p'\alpha}(\nu)} < \infty, \tag{1.13}$$

where p' is the conjugate exponent of p. Then there exists $C > 0$ such that

$$||\mathcal{M}_{\alpha,\mathfrak{n}} f||_{L^p(\nu)} \leq C||f||_{L^p(\nu)} \tag{1.14}$$

for all $f \in L^p(\nu)$. Furthermore, the limit $\lim_{k\to\infty} \mathcal{A}_{n_k,\alpha}$ exists almost everywhere and in the $L^p(\nu)$-norm. (In particular, the conclusions of this theorem hold if $g = 1$ and τ is an invertible measure preserving transformation.)

Remark 1.10. On the one hand, notice that (1.13) is an assumption similar to the one in Theorem 1.2. On the other hand, we point out that (1.13) is equivalent to

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} S^k \right\|_{L^{\frac{p\alpha}{p\alpha-p+1}}(\nu)} < \infty, \tag{1.15}$$

where $Sf(x) = g_1^{1/\alpha}(x) J_1^{1-1/\alpha}(x) f(\tau x)$.

Remark 1.11. Theorem 1.9 can be proved assuming that that $T : L^p(\nu) \to L^p(\nu)$, $1 < p < \infty$, is a bounded invertible linear operator such that T and T^{-1} are positive (that is, $f \geq 0$ implies $Tf \geq 0$ and $T^{-1}f \geq 0$). We recall (see [12]) that, in this case, T is a separation-preserving operator on $L^p(\nu)$ and, moreover, for each $j \in \mathbb{Z}$, there exist a positive measurable function g_j and a σ-algebra automorphism Φ_j mapping measurable functions onto measurable functions, such that: (i) $f = h$ a.e. implies $\Phi_j f = \Phi_j h$ a.e.; (ii) for every $f \in L^p(\nu)$, $T^j f = g_j \Phi_j f$; (iii) Φ_j preserves the ν-a.e. convergence of sequences of measurable functions. It follows from these properties that Φ_j is, in particular, a positive linear transformation on the measurable functions (modulo ν), that the sequences $\{g_j\}_{j=-\infty}^{\infty}$ are uniquely determined, and that for $j \in \mathbb{Z}$, f measurable, and $0 < s < +\infty$, we have $|\Phi_j(f)|^s = \Phi_j(|f|^s)$. By the Radon-Nikodym Theorem, there exists a unique sequence $\{J_j\}_{j=-\infty}^{\infty}$ uniquely determined (modulo ν) such that for each $j \in \mathbb{Z}$, $J_j > 0$ on X, and $\int_X f d\nu = \int_X J_j \Phi_j(f) d\nu$, for all $f \in L^1(\nu)$. Then we can see that the proof of Theorem 1.9 can be easily adapted to this situation

By using transference arguments, the study of the maximal operator $\mathcal{M}_{\alpha,\mathfrak{n}}$ can be reduced to the study of the ergodic Cesàro-α maximal operator over the integer numbers associated to the transformation $\tau(i) = i + 1$, which is defined by

$$m^+_{\alpha,\mathfrak{n}} a(i) = \sup_{k\in\mathbb{N}} \frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^{\alpha-1}_{n_k-j} |a(i+j)|, \tag{1.16}$$

where $a : \mathbb{Z} \to \mathbb{R}$ is any function (any sequence of real numbers). In particular, we shall need to study the boundedness of $m^+_{\alpha,\mathfrak{n}}$ on weighted ℓ^p spaces. It is convenient to observe that, by the properties of the Cesàro numbers,

$$\frac{1}{C} m^+_{\alpha,\mathfrak{n}} a(i) \leq \sup_{k\in\mathbb{N}} \frac{1}{(n_k+1)^\alpha} \sum_{j=0}^{n_k} (n_k - j + 1)^{\alpha-1} |a(i+j)| \leq C m^+_{\alpha,\mathfrak{n}} a(i), \tag{1.17}$$

where C is independent of a and i. We notice that $m^+_{\alpha,\mathfrak{n}}$ has a continuous counterpart, the lacunary Cesàro-α maximal function associated to $\mathfrak{E} = \{\varepsilon_k\}_{k\in\mathbb{Z}}$, defined for functions on the real line by

$$M^+_{\alpha,\mathfrak{E}} f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{\varepsilon_k^\alpha} \int_x^{x+\varepsilon_k} (x + \varepsilon_k - t)^{\alpha-1} |f(t)|\, dt, \tag{1.18}$$

where $\mathfrak{E}$ is a ρ–lacunary sequence of positive numbers, that is, $1 < \rho \le \varepsilon_{k+1}/\varepsilon_k \le \rho^2$ for all $k \in \mathbb{Z}$ (it is not necessary to ask for $\varepsilon_1 = 1$). The results for $M^+_{\alpha,\mathfrak{E}}$ and $m^+_{\alpha,\mathfrak{n}}$ are essentially the same, although the proofs may require different details.

The article is organized in the following way: The results for $m^+_{\alpha,\mathfrak{n}}$ (and $M^+_{\alpha,\mathfrak{E}}$) appear in §2. The proofs of Theorem 1.6 and Corollary 1.8 are in §3. Theorem 1.9 is proved in §4. In §5 we give examples that show that the condition in Theorem 1.6 holds while the condition in Corollary 1.8 is not satisfied; we also provide similar examples related to Theorem 2.6 and Corollary 2.7. Finally, some properties established in §2 are proved in §6.

Throughout this paper, α is a number such that $0 < \alpha \le 1$, if $1 < p < \infty$ then p' denotes its conjugate exponent, i.e., $1/p + 1/p' = 1$, and the letter C means a positive constant non necessarily the same at each occurrence.

2. The boundedness of $m^+_{\alpha,\mathfrak{n}}$ and $M^+_{\alpha,\mathfrak{E}}$ in weighted spaces

We are interested in the boundedness of the lacunary maximal operators defined in (1.16) and in (1.18). Our starting points are the whole maximal operators defined as follows: given a function $a : \mathbb{Z} \to \mathbb{R}$ and a measurable function on $f : \mathbb{R} \to \mathbb{R}$ we define the Cesàro maximal functions $m^+_\alpha a$ and $M^+_\alpha f$ by

$$m^+_\alpha a(i) = \sup_{n\ge 0} \frac{1}{A^\alpha_n} \sum_{j=0}^{n} A^{\alpha-1}_{n-j} |a(i+j)|, \quad i \in \mathbb{Z} \tag{2.1}$$

and

$$M^+_\alpha f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon^\alpha} \int_x^{x+\varepsilon} (x+\varepsilon-t)^{\alpha-1} |f(t)|\, dy, \quad x \in \mathbb{R}. \tag{2.2}$$

When $\alpha = 1$, the operators are denoted simply by m^+ and M^+ and they are, respectively, the discrete and the continuous version of the one-sided Hardy-Littlewood maximal operator. The operators m^-_α, m^-, M^-_α and M^- are defined in analogous way, reversing the orientation in the integers and in the real line.

The operator m^+_α is bounded in ℓ^p for $p > 1/\alpha$ and it is of weak type $(1,1)$ for $\alpha = 1$ with respect to the counting measure. For $0 < \alpha < 1$ it is of restricted weak type $(1/\alpha, 1/\alpha)$ but it is not of weak type $(1/\alpha, 1/\alpha)$. Analogous results hold for M^+_α in the setting of the real line with the Lebesgue measure (see [11]). Moreover, the characterizations of the boundedness of m^+_α and M^+_α in weighted spaces are known. To state the results we need to introduce some definitions.

Definition 2.1. Let $1 \le p < \infty$. Let w be a nonnegative function on the integers. We shall say that $w \in A^+_{p,\alpha}(\mathbb{Z})$ if there exists a constant $C > 0$ such that

(i) if $p = 1$,

$$m^-_\alpha w(i) = \sup_{n\ge 0} \frac{1}{A^\alpha_n} \sum_{j=i-n}^{i} A^{\alpha-1}_{j-(i-n)} w(j) \le C w(i), \quad \text{for all } i \in \mathbb{Z}; \tag{2.3}$$

(ii) if $1 < p < \infty$,

$$\left(\sum_{j=r}^{s} w(j)\right)^{1/p} \left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{1/p'} \leq C A_{k-r}^{\alpha}, \tag{2.4}$$

for all $r, s, k \in \mathbb{Z}$ with $r \leq s \leq k$.

The continuous version of these definitions are the following.

Definition 2.2. Let $1 \leq p < \infty$. Let w be a nonnegative function on the real line. We shall say that $w \in A_{p,\alpha}^{+}(\mathbb{R})$ if there exists a constant $C > 0$ such that

(i) if $p = 1$,

$$M_{\alpha}^{-} w(x) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon^{\alpha}} \int_{x-\varepsilon}^{x} (y - (x - \varepsilon))^{\alpha - 1} w(y)\, dy \leq C w(x), \quad \text{a.e.} \quad x \in \mathbb{R}; \tag{2.5}$$

(ii) if $1 < p < \infty$,

$$\left(\int_{a}^{b} w(t)\, dt\right)^{1/p} \left(\int_{b}^{c} w^{1-p'}(t)(c-t)^{(\alpha-1)p'}\, dt\right)^{1/p'} \leq C(c-a)^{\alpha}, \tag{2.6}$$

for all $a, b, c \in \mathbb{R}$ with $a < b < c$.

The classes $A_{p,\alpha}^{-}$ are defined in analogous way, reversing the orientation in the integers and in the real line. Notice that $A_{p,1}^{+}(\mathbb{R})$ is the one-sided Muckenhoupt's class $A_{p}^{+}(\mathbb{R})$ (or Sawyer's class [17]) which characterizes the boundedness of M^{+} in weighted spaces.

The characterizations of the boundednes of m_{α}^{+} in weighted spaces are collected in the following theorem.

Theorem 2.3. *Let $0 < \alpha \leq 1$ and let w be a nonnegative function on the integers.*

(a) *Let $p > \frac{1}{\alpha}$. There exists $C > 0$ such that*

$$\sum_{i \in \mathbb{Z}} [m_{\alpha}^{+} a(i)]^{p} w(i) \leq C \sum_{i \in \mathbb{Z}} |a(i)|^{p} w(i),$$

for all functions $a \in \ell^{p}(w)$ if and only if $w \in A_{p,\alpha}^{+}(\mathbb{Z})$.

(b) *m_{α}^{+} is of restricted weak type $(1/\alpha, 1/\alpha)$, that is, m_{α}^{+} applies the Lorentz space $\ell_{1/\alpha,1}(w)$ into the Lorentz-space $\ell_{1/\alpha,\infty}(w)$ if and only if $w \in A_{1}^{+}(\mathbb{Z})$.*

The continuous version of this result, that is, the corresponding characterization of the boundedness of M_{α}^{+} in $\mathbb{R}$ with the measure $w(x)\, dx$ can be found in [15] and it is easily stated changing the boundedness in $\ell^{p}(w)$ by the boundedness in $L^{p}(w(x)\, dx)$ and the conditions $A_{p,\alpha}^{+}(\mathbb{Z})$ by the conditions $A_{p,\alpha}^{+}(\mathbb{R})$. The proofs in [15] can be easily adapted to the discrete setting of the integers. A proof of statement (a) in the discrete setting can be found in [16, see Lemma 2 and Theorem 3] as a particular case of the general results obtained in that paper.

All the results that we shall present in this section have its continuous counterpart. We shall not state them explicitly or we shall not make any comment unless it is necessary for some proof or because of some particular difference between the discrete and the continuous case.

It is well known that, as in the continuous case, $A_p^+(\mathbb{Z})$ has the following symmetric property: if $1 < p < \infty$ then $w \in A_p^+(\mathbb{Z})$ if and only if $\sigma = w^{1-p'} \in A_{p'}^-(\mathbb{Z})$, where p' is the conjugate exponent of p. There is no such a symmetric property when $\alpha < 1$. However, the classes $A_{p,\alpha}^+(\mathbb{Z})$ still have similarities to one-sided Muckenhoupt's class $A_p^+(\mathbb{Z})$ but the proofs are more subtle, possibly due to the lack of symmetry in the $A_{p,\alpha}^+(\mathbb{Z})$. Some of the properties of these classes are collected in the next proposition.

Proposition 2.4. *Let $0 < \alpha \leq 1$, $\frac{1}{\alpha} < p < \infty$ and let w be a nonnegative function on the integers.*

(a) *If $w \in A_{p,\alpha}^+(\mathbb{Z})$ then there exists q such that $q < p$ and $w \in A_{q,\alpha}^+(\mathbb{Z})$.*
(b) *If $w \in A_{p,\alpha}^+(\mathbb{Z})$ then there exists $r > 1$, such that $w^r \in A_{p,\alpha}^+(\mathbb{Z})$.*
(c) *If $w \in A_{1,\alpha}^+(\mathbb{Z})$ then there exists $r > 1$, such that $w^r \in A_{1,\alpha}^+(\mathbb{Z})$.*
(d) *If $w^{1/\alpha} \in A_1^+(\mathbb{Z})$ then $w \in A_{1,\alpha}^+(\mathbb{Z})$.*
(e) *If $w \in A_{p\alpha}^+(\mathbb{Z})$ then $w \in A_{p,\alpha}^+(\mathbb{Z})$.*

Properties (a) and (e) were proved in [15] in the continuous setting, in the real line, and their proofs are easily adapted to the discrete setting. Property (d) is a consequence of Hölder's inequality and the well-known property of $A_1^+(\mathbb{Z})$ weights: if $u \in A_1^+(\mathbb{Z})$ then $u^r \in A_1^+(\mathbb{Z})$ for some $r > 1$ (this is property (c) for $\alpha = 1$). We shall give a detailed proof of properties (b) and (c) in the last section. The same results hold changing $A_{p,\alpha}^+(\mathbb{Z})$ and $A_p^+(\mathbb{Z})$ classes by $A_{p,\alpha}^-(\mathbb{R})$ and $A_p^-(\mathbb{R})$ classes.

Now we are ready to state the results that we need about the lacunary maximal operator. It follows from Theorem 2.3 and the obvious estimate $m_{\alpha,\mathfrak{n}}^+ \leq m_\alpha^+$ that if $p > \frac{1}{\alpha}$ and $w \in A_{p,\alpha}^+(\mathbb{Z})$ then there exists $C > 0$ such that

$$\sum_{i\in\mathbb{Z}} [m_{\alpha,\mathfrak{n}}^+ a(i)]^p w(i) \leq C \sum_{i\in\mathbb{Z}} |a(i)|^p w(i),$$

for all functions $a \in \ell^p(w)$. Our results give sufficient conditions to obtain the dominated estimates in the bad range $1 \leq p < 1/\alpha$ for the lacunary maximal operator, showing that this operator behaves better than m_α^+. In order to state our first result it is convenient to introduce a definition.

Definition 2.5. Let $0 < \alpha \leq 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence and let w be a nonnegative function on the integers. It is said that w satisfies $A_{1,\alpha,\mathfrak{n}}^+(\mathbb{Z})$, or $w \in A_{1,\alpha,\mathfrak{n}}^+(\mathbb{Z})$, if there exists $C > 0$ such that $m_{\alpha,\mathfrak{n}}^- w(i) \leq C w(i)$, for all $i \in \mathbb{Z}$.

Notice that $w \in A_{1,\alpha,\mathfrak{n}}^+(\mathbb{Z})$ if and only if the lacunary averages

$$r_{n_k,\alpha}^+ a(i) = \frac{1}{A_{n_k}^\alpha} \sum_{j=0}^{n_k} A_{n_k-j}^{\alpha-1} a(i+j)$$

are uniformly bounded in the space $\ell^1(w)$.

Theorem 2.6. *Let $0 < \alpha \leq 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence and let w be a nonnegative function on the integers such that $w^r \in A_{1,\alpha,\mathfrak{n}}^+(\mathbb{Z})$ for some $r > 1$. Then there exists $C > 0$ such that*

$$\sum_{\{i: m_{\alpha,\mathfrak{n}}^+ a(i) > \lambda\}} w(i) \leq \frac{C}{\lambda} \sum_{i\in\mathbb{Z}} |a(i)| w(i),$$

for all $\lambda > 0$ and all functions a on $\mathbb{Z}$.

Using assertion (c) in Proposition 2.4 we have that if $w \in A^+_{1,\alpha}(\mathbb{Z})$ then $w^r \in A^+_{1,\alpha}(\mathbb{Z})$ for some $r > 1$; consequently, $w^r \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ for any ρ-lacunary sequence $\mathfrak{n}$ since it is obvious that $m^-_{\alpha,\mathfrak{n}} \le m^-_\alpha$. Therefore, by Theorem 2.6, we have the following corollary.

Corollary 2.7. Let $0 < \alpha \le 1$ and let w be a nonnegative function on the integers such that $w \in A^+_{1,\alpha}(\mathbb{Z})$. Then there exists $C > 0$ such that

$$\sum_{\{i: m^+_{\alpha,\mathfrak{n}} a(i) > \lambda\}} w(i) \le \frac{C}{\lambda} \sum_{i\in\mathbb{Z}} |a(i)| w(i),$$

for all $\lambda > 0$, all functions a on $\mathbb{Z}$ and all ρ-lacunary sequences. (In particular, by Proposition 2.4, if $w^{1/\alpha} \in A^+_1(\mathbb{Z})$ then $w \in A^+_{1,\alpha}(\mathbb{Z})$ and, consequently, the conclusion of the corollary holds.)

We can provide examples showing that the assumption on the weight in Theorem 2.6 for all ρ lacunary sequences is certainly weaker that the corresponding one in Corollary 2.7. We may wonder whether the weak type inequality is true assuming only that w satisfies $A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$. That is an open question. Obviously, we do not know either if for all weights w satisfying $A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ there exists $r > 1$ such that $w^r \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$.

Theorem 2.8. *Let $0 < \alpha \le 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence and $1 < q < p$. Let w be a nonnegative function on the integers and let $\sigma_q = w^{1-q'}$. Assume that there exist $r > 1$ and $C > 0$ such that*

$$\sum_{i\in\mathbb{Z}} [m^-_{\alpha,\mathfrak{n}} |a|^r(i)]^{\frac{q'}{r}} \sigma_q(i) \le C \sum_{i\in\mathbb{Z}} |a(i)|^{q'} \sigma_q(i)$$

for all functions $a \in \ell^{q'}(\sigma_q)$. Then there exists $C > 0$ such that

$$\sum_{\{i: m^+_{\alpha,\mathfrak{n}} a(i) > \lambda\}} w(i) \le \frac{C}{\lambda^q} \sum_{i\in\mathbb{Z}} |a(i)|^q w(i),$$

for all $\lambda > 0$ and all functions $a \in \ell^q(w)$. Consequently, there exists $C > 0$ such that

$$\sum_{i\in\mathbb{Z}} [m^+_{\alpha,\mathfrak{n}} a(i)]^p w(i) \le C^p \sum_{i\in\mathbb{Z}} |a(i)|^p w(i),$$

for all functions $a \in \ell^p(w)$.

Using Proposition 2.4 and the above theorem we have the following corollary.

Corollary 2.9. Let $0 < \alpha \le 1$. If $1 < p < \frac{1}{1-\alpha}$ and $\sigma = w^{1-p'} \in A^-_{p',\alpha}(\mathbb{Z})$, then

$$\sum_{i\in\mathbb{Z}} [m^+_{\alpha,\mathfrak{n}} a(i)]^p w(i) \le C^p \sum_{i\in\mathbb{Z}} |a(i)|^p w(i),$$

for all functions $a \in \ell^p(w)$ and all ρ-lacunary sequences $\mathfrak{n}$. In particular the result holds if $\sigma = w^{1-p'} \in A^-_{p'\alpha}(\mathbb{Z})$ (see (e) in Proposition 2.4).

2.1. **Proof of Theorem 2.6.** We are going to prove Theorem 2.6 deriving it from its continuous version because the proof is clearer in this case. The continuous version of Theorem 2.6 is the following result.

Theorem 2.10. *Let $\mathfrak{E} = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ a ρ-lacunary sequence and $0 < \alpha \le 1$. Let w be a non-negative measurable function on the real line. If $w^r \in A^+_{1,\alpha,\mathfrak{E}}(\mathbb{R})$, that is,*

$$M^-_{\alpha,\mathfrak{E}} w^r(x) = \sup_{k\in\mathbb{Z}} \frac{1}{\varepsilon_k^\alpha} \int_{x-\varepsilon_k}^{x} (t - x + \varepsilon_k)^{\alpha-1} w^r(t)\, dt \le C w^r(x) \quad a.e., \tag{2.7}$$

for some $r > 1$. Then there exists a constant C such that

$$w(\{x \in \mathbb{R} :\ M^+_{\alpha,\mathfrak{E}} f(x) > \lambda\}) \le \frac{C}{\lambda} \|f\|_{L^1(w)} = \frac{C}{\lambda} \int_{\mathbb{R}} |f(x)| w(x)\, dx,$$

for all $\lambda > 0$ and all functions $f \in L^1(w)$, where $w(E)$ means $\int_E w(x)\, dx$.

(Notice that, as in the discrete case, the assumption on the weight is equivalent to the uniform boundedness in $L^1(w^r)$, for some $r > 1$, of the lacunary averages defined by $R^+_{\varepsilon_k,\alpha} f(x) = \frac{1}{\varepsilon_k^\alpha} \int_x^{x+\varepsilon_k} (x + \varepsilon_k - t)^{\alpha-1} f(t)\, dt$.)

Before proving this theorem, we see that Theorem 2.6 follows from Theorem 2.10.

Proof of Theorem 2.6. First, we see that the operator $M^+_{\alpha,\mathfrak{E}}$ is related to $m^+_{\alpha,\mathfrak{n}}$. On the one hand, given the ρ-lacunary sequence $\mathfrak{n} = \{n_k\}_{k\in\mathbb{N}}$ of positive integers (we recall that $n_1 = 1$) we build a ρ-lacunary sequence $\mathfrak{E} = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ of positive real numbers in the following way: $\varepsilon_k = n_k$ if $k \in \mathbb{N}$ and $\varepsilon_k = \rho^{k-1}$ if $k \le 0$. On the other hand, for any sequence a we consider the function A on the real line given by $A(x) = a([x])$, where $[x]$ is the integer part of x. It is not difficult to show that there exist a constant C such that

$$\begin{aligned} M^-_{\alpha,\mathfrak{E}} A(x) &\le C m^-_{\alpha,\mathfrak{n}} a(i) \quad \text{for all } i \text{ and all } x \in (i, i+1) \\ m^+_{\alpha,\mathfrak{n}} a(i) &\le C M^+_{\alpha,\mathfrak{E}} A(x) \quad \text{for all } i \text{ and all } x \in (i + \tfrac{1}{4}, i + \tfrac{3}{4}). \end{aligned} \tag{2.8}$$

To prove (2.8), it is convenient to use (1.17). Taking into account the inequalities in (2.8), Theorem 2.6 is an easy consequence of Theorem 2.10 as we see in the next lines.

Using the first relation in (2.8), we see that if w^r satisfies $A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ then the function $W^r(x) = w^r([x]) \in A^+_{1,\alpha,\mathfrak{E}}(\mathbb{R})$. By Theorem 2.10, we have

$$W(\{x \in \mathbb{R} :\ M^+_{\alpha,\mathfrak{E}} f(x) > \lambda\}) \le \frac{C}{\lambda} \|f\|_{L^1(W)},$$

for all $\lambda > 0$ and all functions $f \in L^1(W)$. Now take any sequence a and let A be its associated function on the real line, $A(x) = a([x])$. It follows from the second inequality in (2.8) that

$$\begin{aligned}\sum_{\{i\in\mathbb{Z}:m^+_{\alpha,\mathfrak{n}}a(i)>\lambda\}} w(i) &= 2\sum_{\{i\in\mathbb{Z}:m^+_{\alpha,\mathfrak{n}}a(i)>\lambda\}} \int_{i+\frac{1}{4}}^{i+\frac{3}{4}} W(x)\,dx\\ &\le 2\int_{\{x:M^+_{\alpha,\mathfrak{E}}A(x)>\lambda/C\}} W(x)\,dx\\ &\le \frac{C}{\lambda}\int_{\mathbb{R}} A(x)W(x)\,dx = \frac{C}{\lambda}\sum_{i\in\mathbb{Z}}|a(i)|w(i),\end{aligned}$$

as we wished to show. □

2.1.1. *Proof of Theorem 2.10.* First notice that it is enough to prove the theorem for the operator

$$f \longrightarrow \sup_{k\in\mathbb{Z}}\left|\frac{1}{\varepsilon_k^\alpha}\int_x^{x+\varepsilon_k}(x+\varepsilon_k-y)^{\alpha-1}f(y)\,dy\right|.$$

We also denote with $M^+_{\alpha,\mathfrak{E}}$ the above operator. The proof of the theorem follows the lines of the proof of Theorem 1.7 in [4]. Our hypothesis in the present paper is weaker. We write the details of the proof to show clearly where the proof must change.
In what follows, we shall use that w belongs also to $A_1^+(\mathbb{R})$. Moreover, observe that $w^r \in A_1^+(\mathbb{R})$ since $M^-w^r \le CM^-_{0,\mathfrak{E}}w^r \le Cw^r$ a.e..

We begin studying the behavior of $M^+_{\alpha,\mathfrak{E}}$ on the functions of bounded support and average zero.

Lemma 2.11. *Let $0<\alpha\le 1$ and $\mathfrak{E}=\{\varepsilon_k\}_{k\in\mathbb{Z}}$ a ρ-lacunary sequence. Let a be supported on $I=(0,\varepsilon_i)$ and such that $\int_I a = 0$. Assume that w is a weight such that $w^r\in A^+_{1,\alpha,\mathfrak{E}}$ for some $r>1$. Then there exists $C>0$, independent of a, such that*

$$\int_{z<-\varepsilon_{i+\beta}} M^+_{\alpha,\mathfrak{E}}a(z)w(z)\,dz \le C\int_I |a(z)|w(z)\,dz,$$

where β is the smallest positive integer such that $1/\rho+(1/\rho)^\beta\le 1$.

Proof. Let us write

$$\int_{z<-\varepsilon_{i+\beta}} M^+_{\alpha,\mathfrak{E}}a(z)w(z)\,dz = \sum_{m=i+\beta}^{\infty}\int_{-\varepsilon_{m+1}}^{-\varepsilon_m} M^+_{\alpha,\mathfrak{E}}a(z)w(z)\,dz$$

and

$$M^+_{\alpha,\mathfrak{E}}a(z) \le \sum_{k=-\infty}^{\infty}\left|\frac{1}{\varepsilon_k^\alpha}\int_I a(u)(\varepsilon_k+z-u)^{\alpha-1}\chi_{(-\varepsilon_k,0)}(z-u)\,du\right|.$$

Observe that if $z\in(-\varepsilon_{m+1},-\varepsilon_m)$ and $u\in I$ we have that $z-u\in(-\varepsilon_{m+2},-\varepsilon_m)$. Then $z-u\in(-\varepsilon_k,0)$ for all $k\ge m+2$ and $z-u\notin(-\varepsilon_k,0)$ for all $k\le m$.

Therefore, for all $z \in (-\varepsilon_{m+1}, -\varepsilon_m)$,

$$M^+_{\alpha,\mathfrak{E}} a(z) \le \left| \int_I \frac{(\varepsilon_{m+1} + z - u)^{\alpha-1}}{\varepsilon^\alpha_{m+1}} \chi_{(-\varepsilon_{m+1},0)}(z-u) a(u)\, du \right| + \left| \int_I \frac{(\varepsilon_{m+2} + z - u)^{\alpha-1}}{\varepsilon^\alpha_{m+2}} a(u)\, du \right| + \sum_{k=m+3}^{\infty} \left| \int_I \frac{(\varepsilon_k + z - u)^{\alpha-1}}{\varepsilon^\alpha_k} a(u)\, du \right| = A_m(z) + B_m(z) + C_m(z).$$

Now we have to change the proof in [4]. Let $\delta \in (1, \rho)$ to be chosen later. Then

$$\begin{aligned} &\int_{-\varepsilon_{m+1}}^{-\varepsilon_m} A_m(z) w(z)\, dz \\ &\le \int_{-\varepsilon_{m+1}}^{-\varepsilon_{m+1}+\delta^{m-i}\varepsilon_i} w(z) \int_I \frac{(\varepsilon_{m+1} + z - u)^{\alpha-1}}{\varepsilon^\alpha_{m+1}} \chi_{(-\varepsilon_{m+1},0)}(z-u) |a(u)|\, du\, dz \\ &\quad + \int_{-\varepsilon_{m+1}+\delta^{m-i}\varepsilon_i}^{0} w(z) \left| \int_I \frac{(z + \varepsilon_{m+1} - u)^{\alpha-1}}{\varepsilon^\alpha_{m+1}} a(u)\, du \right| dz. \end{aligned}$$

Now, doing a similar split with $B_m(z)$ we get

$$\begin{aligned} &\int_{-\varepsilon_{m+1}}^{-\varepsilon_m} B_m(z) w(z)\, dz \\ &\le \int_{-\varepsilon_{m+2}}^{-\varepsilon_{m+2}+\delta^{m+1-i}\varepsilon_i} w(z) \int_I \frac{(\varepsilon_{m+2} + z - u)^{\alpha-1}}{\varepsilon^\alpha_{m+2}} \chi_{(-\varepsilon_{m+2},0)}(z-u) |a(u)|\, du\, dz \\ &\quad + \int_{-\varepsilon_{m+2}+\delta^{m+1-i}\varepsilon_i}^{0} w(z) \left| \int_I \frac{(z + \varepsilon_{m+2} - u)^{\alpha-1}}{\varepsilon^\alpha_{m+2}} a(u)\, du \right| dz. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{z<-\varepsilon_{i+\beta}} M^+_{\alpha,\mathfrak{E}} a(z) w(z)\, dz \le \sum_{m=i+\beta}^{\infty} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} \left(A_m(z) + B_m(z) + C_m(z)\right) w(z)\, dz \\ &\le 2 \sum_{m=i+\beta}^{\infty} \int_{-\varepsilon_{m+1}}^{-\varepsilon_{m+1}+\delta^{m-i}\varepsilon_i} w(z) \int_0^{\varepsilon_i} \frac{(\varepsilon_{m+1} + z - u)^{\alpha-1}}{\varepsilon^\alpha_{m+1}} \chi_{(-\varepsilon_{m+1},0)}(z-u) |a(u)|\, du\, dz \\ &\quad + 2 \sum_{m=i+\beta}^{\infty} \int_{-\varepsilon_{m+1}+\delta^{m-i}\varepsilon_i}^{0} w(z) \left| \int_0^{\varepsilon_i} \frac{(z + \varepsilon_{m+1} - u)^{\alpha-1}}{\varepsilon^\alpha_{m+1}} a(u)\, du \right| dz \\ &\quad + \sum_{m=i+\beta}^{\infty} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} w(z) \sum_{k=m+3}^{\infty} \left| \int_0^{\varepsilon_i} \frac{(\varepsilon_k + z - u)^{\alpha-1}}{\varepsilon^\alpha_k} a(u)\, du \right| dz = I + II + III. \end{aligned}$$

Now we shall prove that each sum is dominated by $C \int_0^{\varepsilon_i} |a(u)| w(u)\, du$. By Fubini's theorem, Hölder's inequality and the hypothesis on the weight w, we obtain for the first sum the following inequalities:

$$
\begin{aligned}
I \leq 2 & \sum_{m=i+\beta}^{\infty} \int_0^{\varepsilon_i} |a(u)| \frac{1}{\varepsilon_{m+1}^{\alpha}} \left(\int_{-\varepsilon_{m+1}+u}^{u} w^r(z)(\varepsilon_{m+1}+z-u)^{\alpha-1}\, dz \right)^{1/r} \\
& \qquad \times \left(\int_{-\varepsilon_{m+1}+u}^{-\varepsilon_{m+1}+\delta^{m-i}\varepsilon_i} (\varepsilon_{m+1}+z-u)^{\alpha-1}\, dz \right)^{1/r'} du \\
\leq & C \sum_{m=i+\beta}^{\infty} \int_0^{\varepsilon_i} |a(u)| [M^-_{\alpha,\mathfrak{C}} w^r(u)]^{1/r} \frac{1}{\varepsilon_{m+1}^{\alpha/r'}} (\delta^{m-i}\varepsilon_i)^{\alpha/r'}\, du \\
\leq & C \sum_{m=i+\beta}^{\infty} \left(\frac{\delta}{\rho}\right)^{(m-i)\alpha/r'} \int_0^{\varepsilon_i} |a(u)| w(u)\, du \leq C \int_0^{\varepsilon_i} |a(u)| w(u)\, du.
\end{aligned}
$$

We shall estimate now the second sum. First we write II as

$$
2 \sum_{m=i+\beta}^{\infty} \sum_{\ell=m-i}^{\infty} \int_{-\varepsilon_{m+1}+\delta^{\ell}\varepsilon_i}^{-\varepsilon_{m+1}+\delta^{\ell+1}\varepsilon_i} \left| \int_0^{\varepsilon_i} a(u) \frac{(z+\varepsilon_{m+1}-u)^{\alpha-1}}{\varepsilon_{m+1}^{\alpha}}\, du \right| w(z)\chi_{(-\infty,0)}(z)\, dz.
$$

Using that $\int_0^{\varepsilon_i} a = 0$, the mean value theorem and the fact that $z \in (-\varepsilon_{m+1} + \delta^{\ell}\varepsilon_i, -\varepsilon_{m+1} + \delta^{\ell+1}\varepsilon_i)$ we obtain that

$$
\begin{aligned}
\left| \int_0^{\varepsilon_i} a(u)(z+\varepsilon_{m+1}-u)^{\alpha-1}\, du \right| & \leq C \int_0^{\varepsilon_i} |a(u)|(z+\varepsilon_{m+1}-u)^{\alpha-2} u\, du \\
& \leq C \int_0^{\varepsilon_i} |a(u)|((\delta^{\ell}-1)\varepsilon_i)^{\alpha-2}\varepsilon_i\, du.
\end{aligned}
$$

Therefore, since $\ell \geq m-i \geq \beta$ and $\frac{\delta^{\ell}-1}{\delta^{\ell}} \geq 1 - \frac{1}{\delta^{\beta}}$, II is bounded by

$$
C_\delta \sum_{m=i+\beta}^{\infty} \frac{(\varepsilon_i)^{\alpha-1}}{(\varepsilon_{m+1})^{\alpha}} \int_0^{\varepsilon_i} |a(u)|\, du \sum_{\ell=m-i}^{\infty} \delta^{\ell(\alpha-2)} \int_{-\varepsilon_{m+1}+\delta^{\ell}\varepsilon_i}^{-\varepsilon_{m+1}+\delta^{\ell+1}\varepsilon_i} w(z)\chi_{(-\infty,0)}(z)\, dz.
$$

By Hölder's inequality and using that $w^r \in A_1^+$ we get

$$
\begin{aligned}
\int_{-\varepsilon_{m+1}+\delta^{\ell}\varepsilon_i}^{-\varepsilon_{m+1}+\delta^{\ell+1}\varepsilon_i} w(z)\chi_{(-\infty,0)}(z)\, dz & \leq C \left(\int_{-\varepsilon_{m+1}}^{0} w^r \right)^{1/r} (\delta^{\ell}\varepsilon_i)^{1/r'} \\
& \leq C \varepsilon_{m+1}^{1/r} \operatorname{ess\,inf}_{x\in(0,\varepsilon_i)} w(x)\ (\delta^{\ell}\varepsilon_i)^{1/r'}
\end{aligned}
$$

and consequently

$$
II \leq C_\delta \sum_{m=i+\beta}^{\infty} \left(\frac{\varepsilon_i}{\varepsilon_{m+1}} \right)^{\alpha-1+1/r'} \left(\int_0^{\varepsilon_i} |a(u)| w(u)\, du \right) \sum_{\ell=m-i}^{\infty} \delta^{\ell(\alpha-2+1/r')}.
$$

Observe that if r is such that $w^r \in A^+_{1,\alpha,\mathfrak{C}}$ then $w^s \in A^+_{1,\alpha,\mathfrak{C}}$ for all $s \in (1,r)$. Then we can assume that $\alpha - 1 + 1/r' < 0$. So that

$$
II \leq C_\delta \int_0^{\varepsilon_i} |a(u)| w(u)\, du \sum_{m=i+\beta}^{\infty} \left(\frac{\delta^{\alpha-2+1/r'}}{\rho^{2(\alpha-1+1/r')}} \right)^{m-i}.
$$

Then, taking $\delta \in (\rho^{\frac{2(\alpha-1+1/r')}{\alpha-2+1/r'}}, \rho) \subset (1,\rho)$, we get that the above series converges to a constant depending on α, ρ and r. So that $II \leq C \int_0^{\varepsilon_i} |a(u)| w(u)\, du$.

In order to estimate the third sum, we use again that $\int_0^{\varepsilon_i} a = 0$ and the mean value theorem. Then we have

$$\begin{aligned} III &\le C \sum_{m=i+\beta}^{\infty} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} w(z) \sum_{k=m+3}^{\infty} \int_0^{\varepsilon_i} \frac{(\varepsilon_k + z - u)^{\alpha-2}}{\varepsilon_k^{\alpha}} u|a(u)|\, du\, dz \\ &\le C \sum_{m=i+\beta}^{\infty} \sum_{k=m+3}^{\infty} \int_0^{\varepsilon_i} \varepsilon_i |a(u)| \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} \frac{(\varepsilon_k + z - u)^{\alpha-2}}{\varepsilon_k^{\alpha}} w(z)\, dz\, du \end{aligned}$$

Obvious inequalities, (1.7), (1.8) and $w \in A_1^+$ give for almost every $u \in (0, \varepsilon_i)$

$$\begin{aligned} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} (\varepsilon_k + z - u)^{\alpha-2} w(z)\, dz &\le (\varepsilon_k - \varepsilon_{m+1} - \varepsilon_i)^{\alpha-2} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} w(z)\, dz \\ &\le C(\varepsilon_{m+1} + \varepsilon_i)(\varepsilon_{m+3} - \varepsilon_{m+1} - \varepsilon_i)^{\alpha-2} w(u) \le C\varepsilon_{m+3}^{\alpha-1} w(u) \end{aligned}$$

Consequently,

$$\begin{aligned} III &\le C \left(\int_0^{\varepsilon_i} |a(u)| w(u)\, du \right) \varepsilon_i \sum_{m=i+\beta}^{\infty} \varepsilon_{m+3}^{\alpha-1} \sum_{k=m+3}^{\infty} \frac{1}{\varepsilon_k^{\alpha}} \\ &\le C \left(\int_0^{\varepsilon_i} |a(u)| w(u)\, du \right) \varepsilon_i \sum_{m=i+\beta}^{\infty} \frac{1}{\varepsilon_{m+3}} \le C \int_0^{\varepsilon_i} |a(u)| w(u)\, du \end{aligned}$$

□

Corollary 2.12. Let $0 < \alpha \le 1$, $\mathfrak{E} = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ a ρ-lacunary sequence and let w be a weight such that $w^r \in A^+_{1,\alpha,\mathfrak{E}}$ for some $r > 1$. Let a be supported on $I = (x^*, x^* + h)$ and such that $\int_I a = 0$. If $A = \rho^{2(\beta+1)}$ there exists C independent of x^*, h and a, such that

$$\int_{z<x^*-Ah} M^+_{\alpha,\mathfrak{E}} a(z) w(z)\, dz \le C \int_I |a(z)| w(z)\, dz.$$

Proof. First notice that it is sufficient to prove the corollary for $x^* = 0$. Choose i such that $\varepsilon_{i-1} \le h < \varepsilon_i$. Then a is supported on $(0, \varepsilon_i)$ and has integral zero. Furthermore, by (1.7), $-Ah < -\varepsilon_{i+\beta}$ and by the lemma

$$\int_{z<-Ah} M^+_{\alpha,\mathfrak{E}} a(z) w(z)\, dz \le \int_{z<-\varepsilon_{i+\beta}} M^+_{\alpha,\mathfrak{E}} a(z) w(z)\, dz \le C \int_0^h |a(z)| w(z)\, dz.$$

□

Once we have Corollary 2.12 and Theorem 2.3 the proof of Theorem 2.10 is straightforward (see for instance [4] with $M^+_{\alpha,\mathfrak{E}}$ instead of the operator T^*).

2.2. **Proof of Theorem 2.8.** We shall prove that

$$\sum_{\{i: m^+_{\alpha,\mathfrak{n}} a(i) > \lambda\}} w(i) \le \frac{C}{\lambda^q} \sum_{i\in\mathbb{Z}} |a(i)|^q w(i),$$

for all $\lambda > 0$ and all functions $a \in \ell^q(w)$. Then Marcinkiewicz's interpolation theorem gives the second part of the result. The proof of the above inequality will follow the ideas of Rubio de Francia.

The assumption on the weight can be written as

$$\sum_{i\in\mathbb{Z}}[m^-_{\alpha,\mathfrak{n}}|a|^r(i)]^{\frac{q'}{r}}\sigma_q(i)\le C^{q'}\sum_{i\in\mathbb{Z}}|a(i)|^{q'}\sigma_q(i). \tag{2.9}$$

for some constant C. We may assume that the set $\{i : a(i)\neq 0\}$ is finite. Let $O_\lambda=\{i : m^+_{\alpha,\mathfrak{n}}a(i)>\lambda\}$. Then there exists $u\in\ell^{q'}(\sigma_q)$ with $||u||_{\ell^{q'},\sigma_q}=1$ such that $||\chi_{O_\lambda}||_{\ell^q,w}=\sum_{i\in\mathbb{Z}}\chi_{O_\lambda}(i)u(i)$. Notice that given $0\le u\in\ell^{q'}(\sigma_q)$, we can define

$$U(i)=\sum_{j=0}^{\infty}\frac{[(m^-_{\alpha,\mathfrak{n}})^{(j)}u^r(i)]^{1/r}}{(2C)^j},$$

where C is the constant in (2.9) and $(m^-_{\alpha,\mathfrak{n}})^{(j)}$ is the j-th iteration of the maximal operator. It is easy to see that $u\le U$, $||U||_{\ell^{q'},\sigma_q}\le 2C||u||_{\ell^{q'},\sigma_q}$ and $m^-_{\alpha,\mathfrak{n}}U^r(i)\le 2CU^r(i)$, that is, $U^r\in A^+_{1,\alpha,\mathfrak{n}}$. Now, applying Theorem 2.6 and the Hölder inequality we have

$$\begin{aligned}\sum_{\{i:m^+_{\alpha,\mathfrak{n}}a(i)>\lambda\}}w(i)&=||\chi_{O_\lambda}||^q_{\ell^q,w}\le\left(\sum_i\chi_{O_\lambda}(i)U(i)\right)^q=\left(\sum_{\{i:m^+_{\alpha,\mathfrak{n}}a(i)>\lambda\}}U(i)\right)^q\\&\le\frac{C}{\lambda^q}\left(\sum_{i\in\mathbb{Z}}|a(i)|U(i)\right)^q\le\frac{C}{\lambda^q}\sum_{i\in\mathbb{Z}}|a(i)|^q w(i)\left(\sum_{i\in\mathbb{Z}}[U(i)]^{q'}\sigma_q(i)\right)^{q/q'}\\&\le\frac{C}{\lambda^q}\sum_{i\in\mathbb{Z}}|a(i)|^q w(i).\end{aligned}$$

2.3. **Proof of Corollary 2.9.** From Proposition 2.4 if $\sigma=w^{1-p'}\in A^-_{p',\alpha}$ then there exists $r>1$ such that $w^{(1-p')r}\in A^-_{p',\alpha}$. Let q be a number such that $1-q'=(1-p')r$, so that $q'>p'$. Since $A^-_{p',\alpha}\subset A^-_{q',\alpha}$, then $w^{(1-p')r}\in A^-_{q',\alpha}$, i.e., $\sigma_q=w^{1-q'}\in A^-_{q',\alpha}$. By Proposition 2.4, there exists $r>1$ such that $\sigma_q=w^{1-q'}\in A^-_{q'/r,\alpha}$. Using Theorem 2.3 (with m^-_α instead of m^+_α) we have immediately that $m^-_{\alpha,r}a=(m^-_\alpha a^r)^{1/r}$ applies $\ell^{q'}(\sigma_q)$ into $\ell^{q'}(\sigma_q)$. This implies the assumption in Theorem 2.8 for all ρ-lacunary sequences $\mathfrak{n}$ and the corollary follows.

2.4. **Remark.** Notice that the proofs of Theorem 2.8 and Corollary 2.9 in the continuous case are exactly the same.

2.5. **Example.** Let $0<\alpha<1$ and let $\mathfrak{E}=\{\varepsilon_k\}$ a ρ-lacunary sequence. For all r, $1<r<1/\alpha$, there exists a weight w on the real line such that $w^r\in A^+_{1,\alpha,\mathfrak{E}}(\mathbb{R})$ and $w^{1/\alpha}\notin A^+_1(\mathbb{R})$. We follow ideas of Rubio de Francia about factorization of weights.

Let us choose any p, $1<p<\frac{1}{r\alpha}$. $M^-_{\alpha,\mathfrak{E}}$ is bounded in $L^p(dx)$ since it is of weak type $(1,1)$ by Theorem 2.10 and it is obviously of strong type (∞,∞). Let A be a constant such that $||M^-_{\alpha,\mathfrak{E}}(f)||_{L^p(dx)}\le A||f||_{L^p(dx)}$ for all f. Let $0<f\in L^p(dx)$ such that $f^{\frac{1}{r\alpha}}$ is not integrable in $(-1,1)$ and consider the function $F=\sum_{i=0}^\infty(2A)^{-i}(M^-_{\alpha,\mathfrak{E}})^{(i)}f$, where $(M^-_{\alpha,\mathfrak{E}})^{(i)}$ is the i-th iteration of the maximal operator. It is clear that $F\in L^p(dx)$, $F\ge f$ and $M^-_{\alpha,\mathfrak{E}}F\le 2AF$, that is $F\in A^+_{1,\alpha,\mathfrak{E}}(\mathbb{R})$. Let $w=F^{1/r}$. Then $w^r\in A^+_{1,\alpha,\mathfrak{E}}(\mathbb{R})$. However $w^{\frac{1}{\alpha}}$ is not in $A^+_1(\mathbb{R})$ because it is not locally integrable since $w^{\frac{1}{\alpha}}=F^{\frac{1}{r\alpha}}\ge f^{\frac{1}{r\alpha}}$.

3. Proofs of Theorem 1.6 and Corollary 1.8

The next proposition establishes the meaning of the assumption (1.10).

Proposition 3.1. *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, $\mathfrak{n} = \{n_k\}$ a ρ-lacunary sequence, τ an invertible non-singular measurable transformation and $Tf(x) = f(\tau x)$. Let $\widetilde{T_\varepsilon} f(x) = (J_1(x))^{-\varepsilon} f(\tau x)$ for some $\varepsilon > 0$. Then (1.10) holds if and only if for almost every $x \in X$, the functions $w_x(i) = J_i(x)$ on the integers satisfy that $w_x^{1+\varepsilon} \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ with a constant independent of x.*

Proof. The assumption (1.10) for a function f, with $f \geq 0$, $f \in L^1(\nu)$, can be written as

$$\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} \int_X J_j^{-\varepsilon}(x) f(\tau^j x)\, d\nu \leq M_1 \int_X f(x)\, d\nu,$$

for all $k \in \mathbb{N}$, or, equivalently,

$$\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} \int_X J_j^{-1-\varepsilon}(\tau^{-j} x) f(x)\, d\nu \leq M_1 \int_X f(x)\, d\nu.$$

That inequality holds if and only if $\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} J_j^{-1-\varepsilon}(\tau^{-j} x) \leq M_1$ a.e., namely,

$$\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} J_{-j}^{1+\varepsilon}(x) \leq M_1 \quad \text{a.e.}$$

Applying it to $\tau^i x$ for all $i \in \mathbb{Z}$, we get

$$\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} J_{-j}^{1+\varepsilon}(\tau^i x) \leq M_1 \quad \text{a.e.}$$

Multiplying by $J_i^{1+\varepsilon}(x)$ we have

$$\frac{1}{A^\alpha_{n_k}} \sum_{j=0}^{n_k} A^\alpha_{n_k-j} J_{i-j}^{1+\varepsilon}(x) \leq M_1 J_i^{1+\varepsilon}(x) \quad \text{a.e.},$$

as we wished to prove. □

Proof of Theorem 1.6. We start proving the weak type $(1,1)$ inequality (1.11). It suffices to consider nonnegative functions f. Let $L \in \mathbb{N}$, $L > 0$. Let us define

$$\mathcal{M}^L_{\alpha,\mathfrak{n}} f = \sup_{0 \leq k \leq L} \mathcal{A}_{n_k,\alpha} f.$$

Now, given $N \in \mathbb{N}$, by the property of J_i we get that

$$\begin{aligned}\nu(\{x : \mathcal{M}^L_{\alpha,\mathfrak{n}} f(x) > \lambda\}) &= \frac{1}{N+1} \sum_{i=0}^{N} \int_X \chi_{\{\mathcal{M}^L_{\alpha,\mathfrak{n}} f > \lambda\}}(x)\, d\nu(x) \\ &= \frac{1}{N+1} \int_X \sum_{i=0}^{N} \chi_{\{\mathcal{M}^L_{\alpha,\mathfrak{n}} f > \lambda\}}(\tau^i x) J_i(x)\, d\nu(x).\end{aligned}$$

Notice that if $0 \le i \le N$ and $\chi_{\{\mathcal{M}^L_{\alpha,\mathfrak{n}} f>\lambda\}}(\tau^i x) = 1$ then $m^+_{\alpha,\mathfrak{n}}(f_x \chi_{[0,N+n_L]})(i) > \lambda$, where $f_x(i) = f(\tau^i x)$. Therefore,

$$\nu(\{x : \mathcal{M}^L_{\alpha,\mathfrak{n}} f(x) > \lambda\}) \le \frac{1}{N+1} \int_X \sum_{\{i: m^+_{\alpha,\mathfrak{n}}(f_x \chi_{[0,N+n_L]})(i)>\lambda\}} J_i(x)\, d\nu(x).$$

By Proposition 3.1 and Theorem 2.6 we have that there exists a constant C such that for a.e. x

$$\sum_{\{i: m^+_{\alpha,\mathfrak{n}}(f_x \chi_{[0,N+n_L]})(i)>\lambda\}} J_i(x) \le \frac{C}{\lambda} \sum_{i=0}^{N+n_L} f(\tau^i x) J_i(x).$$

Consequently,

$$\begin{aligned}\nu(\{x : \mathcal{M}^L_{\alpha,\mathfrak{n}} f(x) > \lambda\}) &\le \frac{C}{\lambda(N+1)} \int_X \sum_{i=0}^{N+n_L} f(\tau^i x) J_i(x)\, d\nu(x) \\ &= \frac{C(N+n_L+1)}{\lambda(N+1)} \int_X f(x)\, d\nu(x).\end{aligned}$$

The proof of inequality (1.11) finishes letting N and then L tend to ∞.

In order to finish the proof of the theorem, we only have to show the a.e. convergence of $\mathcal{A}_{n_k,\alpha} f$ for f in a dense set of $L^1(\nu)$. We have already used that, by Proposition 3.1, for almost every $x \in X$, the functions $w_x(i) = J_i(x)$ on the integers satisfy that $w_x^{1+\varepsilon} \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ with a constant independent of x. Since $A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z}) \subset A^+_{1,0,\mathfrak{n}}(\mathbb{Z})$ and $A^+_{1,0,\mathfrak{n}}(\mathbb{Z}) = A^+_1(\mathbb{Z})$ we have, for almost every $x \in X$, $w_x^{1+\varepsilon} \in A^+_1(\mathbb{Z})$ and, consequently, $w_x \in A^+_1(\mathbb{Z})$ with constant independent of x. As it was pointed out in Remark 1.5, this condition is equivalent to (1.5) and, therefore, by Theorem 1.4, the full sequence of averages $\mathcal{A}_{n,\alpha} f$ converges almost everywhere for all $f \in L_{1/\alpha,1}(\nu)$ as $n \to \infty$. In particular, the subsequence of averages $\mathcal{A}_{n_k,\alpha} f$ converges almost everywhere for all $f \in L_{1/\alpha,1}(\nu) \cap L^1(\nu)$ which is dense in $L^1(\nu)$. □

Proof of Corollary 1.8. We notice first that (1.12) implies (is equivalent to) that, for almost every $x \in X$, the functions $w_x(i) = J_i(x)$ on the integers satisfy that $w_x \in A^+_{1,\alpha}$ with a constant independent of x. Then, by assertion (c) in Proposition 2.4, there exists $\varepsilon > 0$ such that $w_x^{1+\varepsilon} \in A^+_{1,\alpha}(\mathbb{Z}) \subset A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ for all ρ-lacunary sequences $\mathfrak{n}$ and with constants independent of x. By Proposition 3.1 we have that (1.10) holds for all ρ-lacunary sequences $\mathfrak{n}$. The corollary follows from Theorem 1.6. □

4. Proof of Theorem 1.9

Applying Theorem 1.1 we see that assumption (1.13) holds if and only if $w_x^{1-p'} \in A^-_{p'\alpha}(\mathbb{Z})$ with a constant independent of x, where $w_x(i) = g_i(x)^{-p} J_i(x)$. Then, as in the proof of Theorem 1.6, using transference arguments together with Corollary 2.9 we obtain the strong type inequality (1.14).

To finish the proof of the theorem, it will suffice to show that the sequence of averages $\mathcal{A}_{n_k,\alpha} f$ converge a.e. First, we notice that $\mathcal{M}_{0,\mathfrak{n}} f \le \mathcal{M}_{\alpha,\mathfrak{n}} f$ and $\mathcal{M} f \le C\mathcal{M}_{0,\mathfrak{n}} f$. Since (1.14) holds, we have that there exists $C > 0$ such that

$$||\mathcal{M} f||_{L^p(\nu)} \le C ||f||_{L^p(\nu)} \tag{4.1}$$

for all $f \in L^p(\nu)$. Then (see [14]) the set of functions $D = \{h + f - Tf : h, f \in L^p(\nu), Th = h, \}$ is dense in $L^p(\nu)$. It will suffice to obtain the a.e. convergence of $\mathcal{A}_{n_k,\alpha} f$ for $f \in D$. Since the convergence is obvious for the invariant functions h, we have only to prove that $\mathcal{A}_{n_k,\alpha}(f - Tf)$ converges for every $f \in L^p(\nu)$. It is not difficult to see [13, p.599] that

$$\begin{aligned}\mathcal{A}_{n_k,\alpha}(f - Tf)(x) &= \frac{A_{n_k}^{\alpha-1}}{A_{n_k}^{\alpha}} f(x) - \frac{T^{n_k+1} f(x)}{A_{n_k}^{\alpha}} + \frac{1-\alpha}{A_{n_k}^{\alpha}} \sum_{i=0}^{n_k} \frac{A_{n_k-i}^{\alpha-1}}{n_k + 1 - i} T^i f(x)\\ &= I_k(x) + II_k(x) + III_k(x).\end{aligned}$$

Since $I_k(x) = \dfrac{\alpha}{\alpha + n_k} f(x)$, we have that $\lim_{k\to\infty} I_k(x) = 0$. In order to see that $II_k(x)$ and $III_k(x)$ converge also to 0 we have to work harder.

We know that, for a.e. x, $w_x^{1-p'} = g_i(x)^{p'} J_i^{1-p'}(x) \in A_{p'\alpha}^-(\mathbb{Z})$ with a constant independent of x. Then there exists $s > 1$ such that, for a.e. x, $w_x^{s(1-p')} \in A_{p'\alpha}^-(\mathbb{Z})$ with a constant independent of x. If $r = ps/(p + s - 1)$, $q = p/r$ and q' is the conjugate exponent of q, our last assertion is equivalent to the following: for a.e. x, $(g_i(x)^r)^{q'} J_i^{1-q'}(x) \in A_{p'\alpha}^-(\mathbb{Z})$ with a constant independent of x. Since $p' < q'$ we have that $A_{p'\alpha}^-(\mathbb{Z}) \subset A_{q'\alpha}^-(\mathbb{Z})$. So, finally,

$$\text{for a.e. } x,\ (g_i(x)^r)^{q'} J_i^{1-q'}(x) \in A_{q'\alpha}^-(\mathbb{Z}) \text{ with a constant independent of } x. \tag{4.2}$$

Let T_r be the operator defined by $T_r\varphi(x) = g^r(x)\varphi(\tau x)$ and let $\mathcal{M}_{\alpha,\mathfrak{n},T_r}$ be the ergodic lacunar maximal operator associated to T_r. By (4.2), we can apply to the operator T_r the part of the theorem that we have already proved. Then we have that there exists $C > 0$ such that $||\mathcal{M}_{\alpha,\mathfrak{n},T_r}\varphi||_{L^q(\nu)} \le C||\varphi||_{L^q(\nu)}$ for all $\varphi \in L^q(\nu)$ (notice that $q < p$). In particular, $\mathcal{M}_{\alpha,\mathfrak{n},T_r}\varphi(x) < +\infty$ for a.e. x and $\varphi \in L^q(\nu)$. Observe that

$$|II_k(x)| \le \left| \frac{(T_r^{n_k}|Tf|^r)^{1/r}}{A_{n_k}^{\alpha}} \right| \le \left(\frac{1}{A_{n_k}^{\alpha}} \right)^{1-1/r} \left(\mathcal{M}_{\alpha,\mathfrak{n},T_r}|Tf|^r(x)\right)^{1/r}.$$

Since $f \in L^p(\nu)$ then $Tf \in L^p(\nu)$ and, consequently, $|Tf|^r \in L^q(\nu)$. Therefore, $\mathcal{M}_{\alpha,\mathfrak{n},T_r}(Tf)^r(x) < +\infty$ a.e.. Taking into account that $\lim_{k\to\infty} \frac{1}{A_{n_k}^{\alpha}} = 0$ and $r > 1$ we have that $\lim_{k\to\infty} II_k(x) = 0$ for a.e. x. Finally, applying Hölder's inequality,

$$\begin{aligned}|III_k(x)| &\le \frac{1-\alpha}{A_{n_k}^{\alpha}} \sum_{i=0}^{n_k} \frac{A_{n_k-i}^{\alpha-1}}{n_k + 1 - i} \left(T_r^i |f|^r(x)\right)^{1/r}\\ &\le \left[\frac{1}{A_{n_k}^{\alpha}} \sum_{i=0}^{n_k} A_{n_k-i}^{\alpha-1} T_r^i |f|^r(x) \right]^{1/r} \left[\frac{1}{A_{n_k}^{\alpha}} \sum_{i=0}^{n_k} A_{n_k-i}^{\alpha-1} \frac{1}{(n_k + 1 - i)^{r'}} \right]^{1/r'}\\ &\le \left(\mathcal{M}_{\alpha,\mathfrak{n},T_r}|f|^r(x)\right)^{1/r} \left[\frac{1}{A_{n_k}^{\alpha}} \sum_{i=0}^{n_k} A_{n_k-i}^{\alpha-1} \frac{1}{(n_k + 1 - i)^{r'}} \right]^{1/r'}.\end{aligned}$$

As before, the first term on the right-hand side is finite a.e.. A simple computation (see [13, p. 600 y 601]) shows that the limit of the second term is zero. Consequently, $\lim_{k\to\infty} III_k(x) = 0$ for a.e. x and we are done.

5. Examples

We start showing that the condition in Theorem 1.6 is weaker than the condition in Corollary 1.8. In the statement of the result we use the notations in Theorem 1.6.

Theorem 5.1. *Let $(X, \mathcal{F}, \mu)$ be a nonatomic finite measure space, $0 < \alpha < 1$ and τ an invertible ergodic measurable transformation which preserves the measure μ. Let T be the operator $Tf(x) = f(\tau x)$. For all ρ-lacunary sequences $\mathfrak{n} = \{n_k\}$ there exists a finite measure ν equivalent to μ such that*

$$\sup_{k\in\mathbb{N}} \left\| \frac{1}{A^{\alpha}_{n_k}} \sum_{j=0}^{n_k} A^{\alpha-1}_{n_k-j} \widetilde{T_\varepsilon}^j f \right\|_{L^1(\nu)} < \infty, \tag{5.1}$$

and

$$\sup_{n\geq 0} \left\| \frac{1}{A^{\alpha}_{n}} \sum_{k=0}^{n} A^{\alpha-1}_{n-k} T^k \right\|_{L^1(\nu)} = +\infty. \tag{5.2}$$

Proof. The measure ν will be of the form $d\nu = w d\mu$ for some positive measurable function w. We point out that such a measure satisfies (5.1) and (5.2) if and only if there exists $C > 0$ such that

$$\mathcal{M}^-_{\alpha,\mathfrak{n}} w^{1+\varepsilon} \leq C w(x)^{1+\varepsilon} \quad \text{a.e.} \tag{5.3}$$

and there is no $C > 0$ such that

$$\mathcal{M}^-_{\alpha} w \leq C w(x) \quad \text{a.e.}, \tag{5.4}$$

where $\mathcal{M}^-_{\alpha,\mathfrak{n}}$ and $\mathcal{M}^-_{\alpha}$ are the lacunary and the classical ergodic maximal operators associated to the operator $T^{-1}f(x) = f(\tau^{-1}x)$. Therefore, the theorem will be proved if we find a function w with those properties.
Let $1 < r < 1/\alpha$. Let $p \in (1, 1/r\alpha)$ and let us choose $g \geq 0$ such that $g \in L^p(\mu)$ and $g \notin L^{1/r\alpha}(\mu)$. Since $g^{1/r} \notin L^{1/\alpha}(\mu)$ we have that $g^{1/r} \notin L_{1/\alpha,1}(\mu)$. It was proved in [5] that there exists $h \geq 0$ equimeasurable with $g^{1/r}$ such that $\mathcal{M}^-_{\alpha} h = +\infty$ a.e. Let us take $F = h^r$. Obviously, $F \in L^p(\mu)$, $F \notin L^{1/r\alpha}(\mu)$ and $\mathcal{M}^-_{\alpha} F^{1/r} = +\infty$ a.e. By using, for instance, Corollary 1.8, the lacunary ergodic maximal operator $\mathcal{M}^-_{\alpha,\mathfrak{n}}$ is bounded in $L^p(\mu)$. Let $K > 0$ such that

$$||\mathcal{M}^-_{\alpha,\mathfrak{n}} f||_{L^p(\mu)} \leq K ||f||_{L^p(\mu)}$$

for all $f \in L^p(\mu)$. Let u be the function

$$u = \sum_{i=0}^{\infty} \frac{1}{2^i K^i} (\mathcal{M}^-_{\alpha,\mathfrak{n}})^{(i)} F,$$

where $(\mathcal{M}^-_{\alpha,\mathfrak{n}})^{(i)}$ is the i-th iteration of $\mathcal{M}^-_{\alpha,\mathfrak{n}}$. Clearly $0 \leq F \leq u \in L^p(\mu)$ and $\mathcal{M}^-_{\alpha,\mathfrak{n}} u \leq 2Ku$ a.e. Let $w = u^{1/r}$. Then we have that

$$\mathcal{M}^-_{\alpha,\mathfrak{n}} w^r \leq 2K w^r \quad \text{a.e.} \quad \text{and} \quad \mathcal{M}^-_{\alpha} w \geq \mathcal{M}^-_{\alpha} F^{1/r} = +\infty \quad \text{a.e.}. \tag{5.5}$$

Now take the measure $\nu = w d\mu$ and $\varepsilon = r - 1$ and the theorem is proved. □

Using the last theorem, we provide examples which show that the assumption on the weight in Theorem 2.6 for all ρ lacunary sequences is certainly weaker than the corresponding one in Corollary 2.7.

Corollary 5.2. Let $0 < \alpha \leq 1$. For all ρ-lacunary sequences $\mathfrak{n} = \{n_k\}$ there exists a nonnegative function w on the integers such that $w^r \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ for some $r > 1$ and $w \notin A^+_{1,\alpha}(\mathbb{Z})$.

Proof. Let w and r be as in the proof of Theorem 5.1 It follows from (5.5) that for a.e. x

$$\mathcal{M}^-_{\alpha,\mathfrak{n}} w^r(\tau^i x) \leq 2K w^r(\tau^i x) \quad \text{and} \quad \mathcal{M}^-_{\alpha} w(\tau^i x) = +\infty \quad \text{for all } i \in \mathbb{Z}. \tag{5.6}$$

Therefore, for a.e. x, the function on the integers $w_x(i) = w(\tau^i x)$ satisfies $w_x^r \in A^+_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ for all ρ lacunary sequences and $w_x \notin A^+_{1,\alpha}(\mathbb{Z})$, and, consequently, for a.e. x, w_x is the function w we were looking for.

□

Finally, we are going to find an example satisfying the assumption in Theorem 2.8. As in Corollary 5.2, we have a positive function u on the integers such that $u^\lambda \in A^-_{1,\alpha,\mathfrak{n}}(\mathbb{Z})$ for some $\lambda > 1$. By Theorem 2.6 (reversing the orientation of the real line), we have that $m^-_{\alpha,\mathfrak{n}}$ applies $\ell^1(w)$ into weak-$\ell^1(w)$. Since it is also of strong type (∞, ∞) then it applies $\ell^s(w)$ into $\ell^s(w)$ for $s > 1$. Given q, $1 < q < p$, and r, $1 < r < q'$, there exists C such that

$$\sum_{i\in\mathbb{Z}} [m^-_{\alpha,\mathfrak{n}} |a|^r(i)]^{\frac{q'}{r}} u(i) \leq C \sum_{i\in\mathbb{Z}} |a(i)|^{q'} u(i).$$

Let $w = u^{1-q}$. Then, the last inequality can be written as

$$\sum_{i\in\mathbb{Z}} [m^-_{\alpha,\mathfrak{n}} |a|^r(i)]^{\frac{q'}{r}} w^{1-q'}(i) \leq C \sum_{i\in\mathbb{Z}} |a(i)|^{q'} w^{1-q'}(i),$$

which is the assumption in Theorem 2.8. We do not know whether or not the weight w can be chosen such that $w^{1-p'} \notin A^-_{p',\alpha}(\mathbb{Z})$ (see Corollary 2.9).

6. Proof of assertions (b) and (c) in Proposition 2.4

The key to prove assertions (b) and (c) in Proposition 2.4 is a characterization of $A^+_{p,\alpha}$ classes in terms of the one-sided A_p classes associated to a measure ν. We dedicate the next subsection to these classes.

6.1. One-sided A_p classes with respect to a measure on $\mathbb{Z}$.

We start with the definition of these classes.

Definition 6.1. Let $\nu = \{\nu(j)\}_{j\in\mathbb{Z}}$ be a sequence of nonnegative numbers. We say the sequence $w \in A^+_p(\nu)$ if there exists a constant C such that

(i) If $p = 1$

$$\sup_{n\in\mathbb{Z}, n\geq 0} \frac{\sum_{j=i-n}^{i} w(j)\nu(j)}{\sum_{j=i-n}^{i} \nu(j)} \leq C w(i), \quad \text{for all } i \in \mathbb{Z} \text{ such that } \nu(i) > 0. \tag{6.1}$$

(ii) If $1 < p < \infty$

$$\left(\sum_{j=r}^{s} w(j)\nu(j)\right)^{1/p} \left(\sum_{j=s}^{k} w^{1-p'}(j)\nu(j)\right)^{1/p'} \leq C \sum_{j=r}^{k} \nu(j), \tag{6.2}$$

for all $r, s, k \in \mathbb{Z}$ with $r \leq s \leq k$.

These classes are the good weights for the maximal operator

$$m_\nu^+(f)(\ell) = \sup_{k\in\mathbb{Z}, k\geq 0} \frac{1}{\sum_{j=\ell}^{\ell+k}\nu(j)} \sum_{j=\ell}^{\ell+k} |f(j)|\nu(j),$$

where if $\sum_{j=\ell}^{\ell+k}\nu(j) = 0$ we consider that the corresponding average is zero. It follows from the results in [2] that, for $p > 1$, the maximal operator m_ν^+ is bounded on $\ell^p(w\nu)$ if and only if $w \in A_p^+(\nu)$. The condition $w \in A_1^+(\nu)$ characterizes the weak type $(1,1)$ inequality of m_ν^+ with respect to the measure defined on the integers by the sequence $w\nu$ (see [2]).

We have analogous results for the maximal operator

$$m_\nu^-(f)(\ell) = \sup_{k\in\mathbb{Z}, k\geq 0} \frac{1}{\sum_{j=\ell-k}^{\ell}\nu(j)} \sum_{j=\ell-k}^{\ell} |f(j)|\nu(j),$$

and the classes $A_p^-(\nu)$ which are defined in the obvious way. Notice that $w \in A_p^-(\nu)$ if and only if $\tilde{w} \in A_p^+(\tilde{\nu})$, where $\tilde{w}(i) = w(-i)$ and $\tilde{\nu}(i) = \nu(-i)$. It follows from this fact that if we have a property for $A_p^+(\nu)$ weights then we have the corresponding one for $A_p^-(\nu)$ weights.

For general sequences ν, it is not true that if $w \in A_1^+(\nu)$ then there exists $r > 1$ such that $w^r \in A_1^+(\nu)$ (see [16] and [9]). However, we have the following result.

Proposition 6.2. *Let ν be a positive Borel measure on the integers.*

(i) If $w \in A_1^+(\nu)$ and there is $C > 0$ such that

$$\nu(n+1) \leq C\sum_{j=m}^{n}\nu(j) \text{ for every } m,n\in\mathbb{Z} \text{ with } \sum_{j=m}^{n}\nu(j) > 0, \tag{6.3}$$

then there exists $r > 1$ such that $w^r \in A_1^+(\nu)$.

(ii) If $w \in A_1^-(\nu)$ and there is $C > 0$ such that

$$\nu(n-1) \leq C\sum_{j=n}^{k}\nu(j) \text{ for every } n,k\in\mathbb{Z} \text{ with } \sum_{j=n}^{k}\nu(j) > 0, \tag{6.4}$$

then there exists $r > 1$ such that $w^r \in A_1^-(\nu)$.

This result was essentially stated in [9, Theorem 5.5] for sequences indexed in $\mathbb{N}$. We include a proof since in [9] there is only a sketch of the proof.

Proof. Since the proofs of (i) and (ii) are similar, we shall only prove (i). In order to do this, notice that it is enough to show that exists $\delta > 0$ such that

$$\sum_{j=n}^{m} w^{1+\delta}(j)\nu(j) \leq C\left(\sum_{j=n}^{m} w(j)\nu(j)\right) w^\delta(m), \tag{6.5}$$

for all $m,n\in\mathbb{Z}$ with $\nu(m) > 0$ and where C depends on δ and the $A_1^+(\nu)$ constant of w. For fixed interval $[n,m]$ in $\mathbb{Z}$, let $\lambda > C_1 w(m)$, where C_1 is the $A_1^+(\nu)$ constant of w. Let

$$O_\lambda = \{\ell\in\mathbb{Z} : \ m_\nu^-(w\chi_{[n,m]})(\ell) > \lambda\}.$$

It is known that $O_\lambda = \cup_{i\in\mathbb{N}} I_i$, where I_i are (disjoint) maximal intervals on $\mathbb{Z}$. Let us see that $I_i = [n_i, m_i] \subset [n,m]$. Obviously, $n_i \geq n$. It is easy to show that

$$\frac{1}{\sum_{j=n_i}^{m_i}\nu(j)} \sum_{j=n_i}^{m_i} w(j)\chi_{[n,m]}(j)\nu(j) > \lambda.$$

Then, if $m \le m_i$, since $\lambda > C_1 w(m)$ we get that

$$\frac{1}{\sum_{j=n_i}^{m} \nu(j)} \sum_{j=n_i}^{m} w(j)\nu(j) > C_1 w(m),$$

which is a contradiction since $w \in A_1^+(\nu)$. On the other hand, since ν satisfies (6.3) we get that

$$\begin{aligned}\frac{\sum_{j=n_i}^{m_i} w(j)\chi_{[n,m]}(j)\nu(j)}{\sum_{j=n_i}^{m_i} \nu(j)} &\le \frac{\sum_{j=n_i}^{m_i+1} w(j)\chi_{[n,m]}(j)\nu(j)}{\sum_{j=n_i}^{m_i+1} \nu(j)} \frac{\sum_{j=n_i}^{m_i+1} \nu(j)}{\sum_{j=n_i}^{m_i} \nu(j)} \\ &\le \lambda \left(1 + \frac{\nu(\{m_i+1\})}{\sum_{k=n_i}^{m_i} \nu(j)}\right) \le \lambda(1+C).\end{aligned}$$

We also have that

$$\lambda < \frac{\sum_{j=n_i}^{k} w(j)\chi_{[n,m]}(j)\nu(j)}{\sum_{j=n_i}^{k} \nu(j)} \le C_1 w(k),$$

for all $k \in I_i$ with $\nu(k) > 0$. From these inequalities and since λ has been chosen bigger than $C_1 w(m)$, we have

$$\begin{aligned}\sum_{\{j\in[n,m]:w(j)>\lambda\}} w(j)\nu(j) &\le \sum_{j\in O_\lambda} w(j)\nu(j)\chi_{[n,m]}(j) = \sum_i \sum_{j\in I_i} w(j)\nu(j)\chi_{[n,m]}(j) \\ &\le C \sum_i \lambda \sum_{j\in I_i} \nu(j) = C\lambda\nu(\cup_i I_i) \\ &\le C\lambda\nu(\{j \in [n,m] : \ w(j) > \lambda/C_1\}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{C_1 w(m)}^{\infty} \lambda^{\delta-1} \sum_{\{j\in[n,m]: \ w(j)>\lambda\}} w(j)\nu(j) \ d\lambda \\ &\quad\le C \int_{C_1 w(m)}^{\infty} \lambda^{\delta} \nu(\{j \in [n,m] : \ w(j) > \lambda/C_1\}) \ d\lambda \\ &\quad= C \int_{C_1 w(m)}^{\infty} \lambda^{\delta} \sum_{j=n}^{m} \chi_{\{j:w(j)>\lambda/C_1\}}(j)\nu(j) \ d\lambda \\ &\quad= C \sum_{j=n}^{m} \chi_{\{j:w(j)>w(m)\}}(j)\nu(j) \int_{C_1 w(m)}^{C_1 w(j)} \lambda^{\delta} \ d\lambda \\ &\quad\le \frac{CC_1^{\delta+1}}{1+\delta} \sum_{j=n}^{m} w^{1+\delta}(j)\nu(j).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{C_1 w(m)}^{\infty} \lambda^{\delta-1} &\sum_{\{j\in[n,m]:\ w(j)>\lambda\}} w(j)\nu(j)\ d\lambda \\ &= \sum_{j=n}^{m} \chi_{\{j:w(j)>C_1w(m)\}}(j) w(j)\nu(j) \int_{C_1 w(m)}^{w(j)} \lambda^{\delta-1}\ d\lambda \\ &\geq \frac{1}{\delta}\sum_{j=n}^{m} w^{1+\delta}(j)\nu(j) - \frac{C_1^{\delta}}{\delta}\sum_{j=n}^{m} w(j)\nu(j)w(m)^{\delta}.\end{aligned}$$

Then Choosing δ such that $\frac{1}{\delta} - \frac{CC_1^{\delta+1}}{1+\delta} > 0$, we have (6.5). □

Given a sequence $w \in A_p^+(\nu)$, we have that m_ν^+ is bounded on $\ell^p(w\nu)$ and m_ν^- is bounded on $\ell^{p'}(w^{1-p'}\nu)$ [2]. Then, by using well-known factorizations argument (see [8] and [6]) there exist $u \in A_1^+(\nu)$ and $v \in A_1^-(\nu)$ such that $w = uv^{1-p}$. Using this factorization we can extend the above result to $1 < p < \infty$.

Proposition 6.3. *Let $1 < p < \infty$ and let ν be a positive Borel measure on the integers such that ν satisfies (6.3) and (6.4). If $w \in A_p^+(\nu)$ then there exists $r > 1$ such that $w^r \in A_p^+(\nu)$.*

6.2. **Proof of assertions (b) and (c) in Proposition 2.4.** In order to prove these assertions we need to give other characterizations of $A_{p,\alpha}^+$ classes. We start with a first characterization.

Proposition 6.4. *$w \in A_{p,\alpha}^+(\mathbb{Z})$, $1 < p < \infty$, if and only if the inequality (2.4) holds for $s = \frac{r+k}{2}$.*

Proof. We have to prove that the $A_{p,\alpha}^+(\mathbb{Z})$ condition with $s = \frac{r+k}{2}$ implies (2.4) with the only restriction $r \le s \le k$. It suffices to prove (2.4) in the case $r \le s < k$. First, let us assume that $s \le \frac{r+k}{2}$. Let $\bar{r}$ be such that $s = \frac{\bar{r}+k}{2}$. Since $k - r \le k-\bar{r} \le 2(k-r)$, by the properties of the numbers A_k^α, we have that $A_{k-\bar{r}}^\alpha \le CA_{k-r}^\alpha$. Therefore,

$$\begin{aligned}&\left(\sum_{j=r}^{s} w(j)\right)^{1/p} \left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{1/p'} \\ &\le \left(\sum_{j=\bar{r}}^{s} w(j)\right)^{1/p} \left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{1/p'} \le CA_{k-\bar{r}}^\alpha \le CA_{k-r}^\alpha.\end{aligned}$$

If $s > \frac{r+k}{2}$, let us choose the numbers $n_0 = r < n_1 < \cdots, n_N < n_{N+1} = s$ such that $n_{i+1} = \frac{n_i+k}{2}$ for $i = 0,\dots,N-1$ and $s \le \frac{n_N+k}{2}$. Notice that the numbers n_i are not necessarily integers. Let $I_i = \{j \in \mathbb{Z} : n_i \le j < n_{i+1}\}$ for $i = 0,\dots,N-1$ and $I_N = \{j \in \mathbb{Z} : n_N \le j \le n_{N+1}\}$. Let us denote by $\overline{n_i} = \min I_i$ and $\overline{m_i} = \max I_i$.

Then

$$\sum_{j=r}^{s} w(j)\left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{p/p'} = \sum_{i=0}^{N}\sum_{j=\overline{n_i}}^{\overline{m_i}} w(j)\left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{p/p'}$$
$$\leq \sum_{i=0}^{N}\left\{\sum_{j=\overline{n_i}}^{\overline{m_i}} w(j)\left(\sum_{\overline{m_i}}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{p/p'}\right\}.$$

Since $\overline{m_i} \leq n_{i+1} \leq \frac{k+n_i}{2} \leq \frac{k+\overline{n_i}}{2}$ we can apply the case we have already proved ($s \leq \frac{r+k}{2}$). Using property (iii) of the numbers A_j^α, we have $A_j^\alpha \simeq j^\alpha$ for $j \neq 0$. Then

$$\sum_{j=r}^{s} w(j)\left(\sum_{j=s}^{k} w^{1-p'}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{p/p'} \leq C\sum_{i=0}^{N}(A_{k-\overline{n_i}}^{\alpha})^p \leq C\sum_{i=0}^{N}(k-\overline{n_i})^{\alpha p}$$
$$\leq C\sum_{i=0}^{N}(k-n_i)^{\alpha p} = C(k-r)^{\alpha p}\sum_{i=0}^{N}\left(\frac{1}{2^i}\right)^{\alpha p} \leq C(k-r)^{\alpha p} \leq C(A_{k-r}^{\alpha})^p.$$

□

Definition 6.5. For each $N \in \mathbb{Z}$ we define the following two sequences ν_N and $\tilde{\nu}_N$ on $\mathbb{Z}$ by

(1) $\nu_N(j) = 0$, if $j < N$ and $\nu_N(j) = A_{j-N}^{\alpha-1}$ if $j \geq N$.
(2) $\widetilde{\nu}_N(j) = 0$, if $j > N$ and $\widetilde{\nu}_N(j) = A_{N-j}^{\alpha-1}$ if $j \leq N$.

The next proposition shows a useful characterization of the $A_{p,\alpha}^+(\mathbb{Z})$ weights in terms of $\in A_p^+(\nu)$ classes.

Proposition 6.6.

(1) $w \in A_{1,\alpha}^+(\mathbb{Z})$ *if and only if* $w \in A_1^+(\nu_N)$ *uniformly on* N.
(2) $w \in A_{p,\alpha}^+(\mathbb{Z})$, $1 < p < \infty$, *if and only if* $w(\cdot)A_{N-\cdot}^{1-\alpha} \in A_p^+(\widetilde{\nu}_N)$ *uniformly on* N.

The sentence uniformly on N *means that it is possible to have in* (6.1) *(respectively in* (6.2)*) the same constant* C *for all* N.

Proof. The proof of *(2)* can be found in [16, Theorem 3] in this discrete setting and in [15] in the continuous one. For that reason we shall only prove *(1)*. It is easy to see that $w \in A_1^+(\nu_N)$ uniformly on N implies that $w \in A_{1,\alpha}^+(\mathbb{Z})$. Now we prove the converse. Let $i \in \mathbb{Z}$. Notice that, by the definition (6.1), we only need to consider $N \leq i$. If $i - n \leq N \leq i$,

$$\frac{\sum_{j=i-n}^{i} \nu_N(j)\ w(j)}{\sum_{j=i-n}^{i} \nu_N(j)} = \frac{\sum_{j=N}^{i} A_{j-N}^{\alpha-1}\ w(j)}{\sum_{j=N}^{i} A_{j-N}^{\alpha-1}}$$
$$= \frac{1}{A_{i-N}^{\alpha}}\sum_{j=N}^{i} A_{j-N}^{\alpha-1}\ w(j) \leq Cw(i),$$

where C is the constant in the condition $A^+_{1,\alpha}$. If $N < i - n$,

$$\frac{\sum_{j=i-n}^{i} \nu_N(j)\ w(j)}{\sum_{j=i-n}^{i} \nu_N(j)} = \frac{\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1}\ w(j)}{\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1}}$$
$$= \frac{1}{\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1}} \sum_{j=i-n}^{i} \frac{A_{j-N}^{\alpha-1}}{A_{j-(i-n)}^{\alpha-1}} A_{j-(i-n)}^{\alpha-1}\ w(j).$$

Since $g(j) = \left(\frac{j-N+1}{j-(i-n)+1}\right)^{\alpha-1}$ is an increasing function, using property (iii) of the Cesàro numbers, we get that

$$\frac{\sum_{j=i-n}^{i} \nu_N(j)\ w(j)}{\sum_{j=i-n}^{i} \nu_N(j)} \le C \frac{A_n^\alpha}{\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1}} g(i) \left[\frac{1}{A_n^\alpha} \sum_{j=i-n}^{i} A_{j-(i-n)}^{\alpha-1}\ w(j) \right].$$

It is easy to see that

$$\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1} \ge C \sum_{j=i-n}^{i} (j-N+1)^{\alpha-1} \ge C(i-N+1)^{\alpha-1}(n+1).$$

Then

$$\frac{A_n^\alpha}{\sum_{j=i-n}^{i} A_{j-N}^{\alpha-1}} g(i) \le C \frac{(n+1)^\alpha}{(i-N+1)^{\alpha-1}(n+1)} \frac{(i-N+1)^{\alpha-1}}{(n+1)^{\alpha-1}} \le C,$$

and, therefore,

$$\frac{\sum_{j=i-n}^{i} \nu_N(j)\ w(j)}{\sum_{j=i-n}^{i} \nu_N(j)} \le C \frac{\sum_{j=i-n}^{i} A_{j-(i-n)}^{\alpha-1}\ w(j)}{A_n^\alpha} \le C w(i),$$

with a constant independent of N and i, as we wished to prove. □

Notice that the family of measures $\{\nu_N\}$ satisfies (6.3) and (6.4) uniformly on N. Then from Propositions 6.6 and 6.2 we get the following result which is assertion (c) in Proposition 2.4.

Proposition 6.7. *If $w \in A^+_{1,\alpha}(\mathbb{Z})$ then there exists $r > 1$ such that $w^r \in A^+_{1,\alpha}(\mathbb{Z})$.*

Next we prove the analogous of the above proposition for $1 < p < \infty$ which is assertion (b) in Proposition 2.4.

Proposition 6.8. *Let $1 < p < \infty$. If $w \in A^+_{p,\alpha}(\mathbb{Z})$ then there exists $r > 1$ such that $w^r \in A^+_{p,\alpha}(\mathbb{Z})$.*

Proof. From Proposition 6.6 we have that $w \in A^+_{p,\alpha}(\mathbb{Z})$ if and only if $w(\cdot)A^{1-\alpha}_{N-\cdot} \in A^+_p(\widetilde{\nu}_N)$, with constants uniformly on N. On the other hand, since the measures $\{\widetilde{\nu}_N\}$ satisfy (6.3) and (6.4) uniformly in N, it follows from Proposition 6.3 that there exists $r > 1$ such that $w^r(\cdot)(A^{1-\alpha}_{N-\cdot})^r \in A^+_p(\widetilde{\nu}_N)$ uniformly on N. From this fact, taking $\ell \le s \le k$, $s = \frac{\ell+k}{2}$, and $N = k$ we get that

$$\left(\sum_{j=\ell}^{s} w^r(j)(A^{1-\alpha}_{k-j})^r A^{\alpha-1}_{k-j}\right)^{1/p} \left(\sum_{j=s}^{k} w^{r(1-p')}(j)(A^{1-\alpha}_{k-j})^{r(1-p')} A^{\alpha-1}_{k-j}\right)^{1/p'} \le C A^\alpha_{k-\ell}.$$

Since for every $\ell \leq j \leq s$

$$\begin{aligned}(A_{k-j}^{1-\alpha})^r A_{k-j}^{\alpha-1} &\simeq (k-j+1)^{(\alpha-1)(1-r)} \\ &\geq (k-s+1)^{(\alpha-1)(1-r)} \simeq (k-\ell+1)^{(\alpha-1)(1-r)},\end{aligned}$$

and for every $s \leq j \leq k$

$$\begin{aligned}(A_{k-j}^{1-\alpha})^{r(1-p')} A_{k-j}^{\alpha-1} &\simeq (k-j+1)^{r(1-p')(1-\alpha)+\alpha-1} \\ &= (k-j+1)^{(\alpha-1)p'}(k-j+1)^{(\alpha-1)(1-p')(1-r)} \\ &\geq (k-j+1)^{(\alpha-1)p'}(k-\ell+1)^{(\alpha-1)(1-p')(1-r)} \\ &\simeq (A_{k-j}^{\alpha-1})^{p'}(k-\ell+1)^{(\alpha-1)(1-p')(1-r)},\end{aligned}$$

we get

$$\begin{aligned}&\left(\sum_{j=\ell}^{s} w^r(j)\right)^{1/p} \left(\sum_{j=s}^{k} w^{r(1-p')}(j)(A_{k-j}^{\alpha-1})^{p'}\right)^{1/p'} \\ &\leq C(k-\ell+1)^{-(\alpha-1)(1-r)/p}(k-\ell+1)^{-(\alpha-1)(1-p')(1-r)/p'} A_{k-\ell}^{\alpha} \leq C A_{k-\ell}^{\alpha}.\end{aligned}$$

Then, by Proposition 6.4, $w^r \in A_{p,\alpha}^{+}(\mathbb{Z})$, as we wished to prove. □

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