

Semigroups of composition operators in analytic Morrey spaces

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Abstract. Analytic Morrey spaces belong to the class of function spaces which, like BMOA, are defined in terms of the degree of oscillation on the boundary of functions analytic in the unit disc. We consider semigroups of composition operators on these spaces and focus on the question of strong continuity. It is shown that these semigroups behave like on BMOA.

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1. Introduction

Let \mathbb{D} be the unit disc in the complex plane \mathbb{C} . A family $(\phi_t)_{t \geq 0}$ of analytic self maps of \mathbb{D} is a *semigroup of functions* if the following conditions hold

- (s1) ϕ_0 is the identity map of \mathbb{D} ,
- (s2) $\phi_s \circ \phi_t = \phi_{s+t}$ for all $s, t \geq 0$,
- (s3) $\phi_t \rightarrow \phi_0$ uniformly on compact subsets of \mathbb{D} as $t \rightarrow 0$.

Each such (ϕ_t) induces a semigroup (C_t) of linear transformations on the Fréchet space $\mathcal{H}(\mathbb{D})$ of all analytic functions on \mathbb{D} , by composition

$$C_t(f)(z) = f(\phi_t(z)), \quad f \in \mathcal{H}(\mathbb{D}),$$

and the following pointwise convergence holds

$$\lim_{t \rightarrow 0^+} f(\phi_t(z)) = f(z), \quad z \in \mathbb{D}.$$

If $(X, \|\cdot\|_X)$ is a Banach space consisting of analytic functions on \mathbb{D} , and (ϕ_t) is a semigroup of functions we say that (ϕ_t) *acts* on X if each composition operator C_t is bounded on X . If in addition

$$\lim_{t \rightarrow 0^+} \|f \circ \phi_t - f\|_X = 0 \tag{1.1}$$

for each $f \in X$ then (C_t) is *strongly continuous* on X . In this case it is interesting to study how operator theoretic properties of the operator semigroup (C_t) relate to function theoretic properties of (ϕ_t) .

The study of composition semigroups on spaces of analytic functions started in [3] where E. Berkson and H. Porta studied the basic properties of semigroups of functions (ϕ_t) , and proved that they induce strongly continuous operator semigroups on Hardy spaces. Other authors studied strong continuity on Bergman spaces A^p , Dirichlet spaces, the spaces $BMOA$ and the Bloch space \mathcal{B} , and their subspaces $VMOA$ and \mathcal{B}_0 . More recently strong continuity was studied on mixed norm spaces $H(p, q, \alpha)$ in [2]. Also in the recent article [1] the authors gave a unified proof of several earlier results, and showed that no nontrivial composition semigroup is strongly continuous on $BMOA$, thus extending similar results known for the space of bounded analytic functions H^∞ and the Bloch space \mathcal{B} .

The purpose of this article is to study strong continuity of composition semigroups on analytic Morrey spaces $H^{2,\lambda}$. These spaces are defined in terms of the oscillation of the boundary function. For $0 < \lambda < 1$ and for arcs I on the circle of length $|I|$, the oscillation is compared to $|I|^\lambda$. The end point $\lambda = 1$ corresponds to $BMOA$. For each $0 < \lambda < 1$, $BMOA \subset H^{2,\lambda} \subset H^2$, and these spaces share several properties of $BMOA$. Their definition and their properties are given in the next section.

For every analytic self map ϕ of the disc, the composition operator $f \rightarrow f \circ \phi$ is a bounded operator on each $H^{2,\lambda}$. We will study strong continuity of composition semigroups (C_t) and will prove that (C_t) is not strongly continuous on $H^{2,\lambda}$ unless the it is trivial.

We will denote constants in various inequalities below by C, C', \dots and their values may change from one step to the next.

2. Background on semigroups and Morrey spaces

2.1. Semigroups of functions and composition operators

If (ϕ_t) is a semigroup of functions then each ϕ_t is univalent. The limit

$$G(z) = \lim_{t \rightarrow 0^+} \frac{\phi_t(z) - z}{t}, \quad z \in \mathbb{D}$$

exists uniformly on compact subsets of \mathbb{D} , it is therefore an analytic function on \mathbb{D} , and is called the infinitesimal generator of (ϕ_t) . The trivial semigroup, $\phi_t(z) \equiv z$ for all t , corresponds to the generator $G \equiv 0$. This trivial case will be ignored. The functional equation

$$G(\phi_t(z)) = \frac{\partial \phi_t(z)}{\partial t} = G(z) \frac{\partial \phi_t(z)}{\partial z}, \quad z \in \mathbb{D}, \quad t \geq 0, \quad (2.1)$$

is satisfied, and the generator G has a unique representation

$$G(z) = (\bar{b}z - 1)(z - b)P(z), \quad z \in \mathbb{D}$$

where $b \in \bar{\mathbb{D}}$ and $P \in \mathcal{H}(\mathbb{D})$ with $\operatorname{Re}(P) \geq 0$ on \mathbb{D} . The pair (b, P) is uniquely determined by (ϕ_t) . Conversely every pair (b, P) with $b \in \bar{\mathbb{D}}$ and $P \in \mathcal{H}(\mathbb{D})$ with $\operatorname{Re}(P) \geq 0$ gives rise to a semigroup of functions. The point b is the *Denjoy-Wolff point* of the semigroup. When $b \in \mathbb{D}$ then b is the common fixed point of all ϕ_t . The semigroup (ϕ_t) may be normalized by composing with automorphisms of the disc, and assume that $b = 0$, and it can be represented as

$$\phi_t(z) = h^{-1}(e^{-ct}h(z)), \quad z \in \mathbb{D}, t \geq 0$$

where h is a univalent function mapping \mathbb{D} onto a spirallike domain Ω , with $h(0) = 0$ and $\operatorname{Re}(c) \geq 0$.

When $b \in \partial\mathbb{D}$ then every orbit $\{\gamma_z(t) = \phi_t(z) : t \geq 0\}$ converge to b as $t \rightarrow \infty$ for each $z \in \mathbb{D}$. In this case b is also a common fixed point in the sense that $\lim_{r \rightarrow 1^-} \phi_t(rb) = b$ for all t . By a normalization it can be assumed that $b = 1$, and (ϕ_t) has a representation

$$\phi_t(z) = h^{-1}(h(z) + ct), \quad z \in \mathbb{D}, t \geq 0$$

where h is a univalent function mapping \mathbb{D} onto a close-to-convex domain Ω with $h(0) = 0$ and $\operatorname{Re}(c) > 0$. The above and other details can be found in [3] and [19].

If (ϕ_t) is a semigroup and X a Banach space of analytic functions on \mathbb{D} such that (ϕ_t) acts on X but the induced operator semigroup (C_t) is not strongly continuous on X , it may still happen that (C_t) is strongly continuous on a nontrivial proper subspace of X . Thus we define

$$[\phi_t, X] = \{f \in X : \|f \circ \phi_t - f\|_X \rightarrow 0 \text{ as } t \rightarrow 0\}.$$

A triangle inequality shows that if $\sup_{0 \leq t \leq 1} \|C_t\|_X < \infty$ then $[\phi_t, X]$ is a closed subspace of X which is maximal with respect to the strong continuity requirement. If in addition X contains the constant functions then

$$[\phi_t, X] = \overline{\{f \in X : Gf' \in X\}} \quad (2.2)$$

where G is the generator of (ϕ_t) , see [5]. Note that in this notation, $[\phi_t, X] = X$ means that (ϕ_t) induces a semigroup of bounded composition operators, which is strongly continuous on X . If the space X contains H^∞ and X_0 is the closure of polynomials in X , then for any semigroup (ϕ_t) we have

$$X_0 \subseteq [\phi_t, X],$$

[1, Corollary 1.3]. This in particular implies that if $H^\infty \subset X$ and polynomials are dense in X then for every (ϕ_t) the induced semigroup (C_t) is strongly continuous on X . This is the case for Hardy spaces H^p , $1 \leq p < \infty$, [3] and the Bergman spaces A^p , $1 \leq p < \infty$, [17]. The above also applies when $X = BMOA$ and implies that for every (ϕ_t) the induced (C_t) is strongly continuous on $BMOA_0 = VMOA$.

D. Sarason [16] proved that $VMOA$ consists of those f in $BMOA$ that can be approximated by their rotations inside $BMOA$. This translates in our

language to that for the rotation semigroup $\phi_t(z) = e^{it}z$ we have $VMOA = [\phi_t, BMOA]$. The same is true for the dilation semigroup $\phi_t(z) = e^{-t}z$, and there other semigroups (ϕ_t) , different from rotations and dilations, whose maximal subspace of strong continuity coincides with VMOA; such semigroups were studied in [4]. There are analogous results for the Bloch space [5].

Additional information for strong continuity of (C_t) can be found in [18] for the Dirichlet space, in [9] for weighted Dirichlet spaces, in [21] for Q_p spaces, in [1] and [8] for the disc algebra, and in [2] for mixed norm spaces.

2.2. Analytic Morrey spaces

Let $L^2(\mathbb{T})$ denote the Hilbert space of square integrable functions on the unit circle \mathbb{T} , and H^2 the Hardy space of all analytic functions f on \mathbb{D} such that

$$\|f\|_{H^2}^2 = \sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

For every $f \in H^2$ the radial limits $\tilde{f}(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exist a.e. on \mathbb{T} and the boundary function \tilde{f} is in $L^2(\mathbb{T})$ with $\|\tilde{f}\|_{L^2(\mathbb{T})} = \|f\|_{H^2}$. In addition the map $f \rightarrow \tilde{f}$ is an injection, and identifies H^2 as a closed subspace of $L^2(\mathbb{T})$. For an arc $I \subset \mathbb{T}$ of normalized length $|I| = \frac{1}{2\pi} \int_I d\theta$ and for $f \in L^2(\mathbb{T})$ let

$$f_I = \frac{1}{|I|} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$$

be the average of f over I . The quantity

$$I(f) = \int_I |f(e^{i\theta}) - f_I|^2 \frac{d\theta}{2\pi} = \int_I |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} - |I| |f_I|^2 \quad (2.3)$$

converges to 0 as $|I| \rightarrow 0$, and the rate of the convergence depends on the degree of oscillation of f around its average f_I . Given $0 \leq \lambda \leq 1$ we may isolate those $f \in L^2(\mathbb{T})$ for which $I(f)/|I|^\lambda$ stays bounded or tends to 0 as $|I| \rightarrow 0$, and thus define the space

$$L^{2,\lambda} = \left\{ f \in L^2(\mathbb{T}) : \sup_{I \subset \mathbb{T}} \frac{1}{|I|^\lambda} \int_I |f(e^{i\theta}) - f_I|^2 \frac{d\theta}{2\pi} < \infty \right\}, \quad (2.4)$$

and its subspace

$$L_0^{2,\lambda} = \left\{ f \in L^2(\mathbb{T}) : \lim_{|I| \rightarrow 0} \frac{1}{|I|^\lambda} \int_I |f(e^{i\theta}) - f_I|^2 \frac{d\theta}{2\pi} = 0 \right\}. \quad (2.5)$$

These are the Morrey spaces. They were introduced in connection with regularity of solutions of partial differential equations, see [14], [6], [7]. They are linear spaces complete under the seminorm

$$p_1(f) = \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|^\lambda} \int_I |f(e^{i\theta}) - f_I|^2 d\theta \right)^{1/2}. \quad (2.6)$$

For $\lambda = 1$ these spaces are BMO and VMO, the spaces of bounded and vanishing mean oscillation respectively. For $\lambda = 0$ both spaces coincide with $L^2(\mathbb{T})$.

Using the identification of H^2 as a closed subspace of $L^2(\mathbb{T})$, the analytic Morrey spaces are

$$H^{2,\lambda} = L^{2,\lambda} \cap H^2, \quad H_0^{2,\lambda} = L_0^{2,\lambda} \cap H^2,$$

that is, they consist of those analytic functions in H^2 whose boundary values belong to $L^{2,\lambda}$, respectively to $L_0^{2,\lambda}$. The restriction of (2.6) is then a seminorm on $H^{2,\lambda}$. A second seminorm on $H^{2,\lambda}$ is

$$p_2(f) = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-\lambda}{2}} \|f \circ \phi_a - f(a)\|_{H^2}, \quad (2.7)$$

where $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$, $|a| < 1$, are the Möbius automorphism of \mathbb{D} and this seminorm is equivalent to p_1 , see [23]. The space $H_0^{2,\lambda}$ contains those f for which

$$\lim_{|a| \rightarrow 1} (1 - |a|^2)^{\frac{1-\lambda}{2}} \|f \circ \phi_a - f(a)\|_{H^2} = 0.$$

A third equivalent seminorm is

$$p_3(f) = \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1 - |z|) dA(z) \right)^{1/2} \quad (2.8)$$

where $S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, z/|z| \in I\}$ is the Carleson box based on the arc $I \subset \mathbb{T}$ and $dA(z)$ is the normalized Lebesgue area measure, see [11]. Functions in $H_0^{2,\lambda}$ are then characterized by the condition

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1 - |z|) dA(z) = 0.$$

Each of the above seminorms becomes a norm on $H^{2,\lambda}$ by adding $|f(0)|$ to it. We will write $\|f\|_{2,\lambda}$ for each of the three equivalent norms, thus ignoring the involved constants.

We list below some properties that are needed later. For $0 < \lambda \leq 1$, $H_0^{2,\lambda}$ is the closure of the polynomials in $H^{2,\lambda}$, and for $0 < \lambda < \mu < 1$,

$$H^2 \supset H^{2,\lambda} \supset H_0^{2,\lambda} \supset H^{2,\mu} \supset H_0^{2,\mu} \supset BMOA \supset VMOA.$$

Further $H^{2,\lambda}$ contains the Hardy space $H^{\frac{2}{1-\lambda}}$, [12] and for each $f \in H^{2,\lambda}$

$$|f(z)| \leq \frac{C \|f\|_{2,\lambda}}{(1 - |z|)^{\frac{1-\lambda}{2}}}, \quad z \in \mathbb{D}. \quad (2.9)$$

The function $f(z) = (1 - z)^{-\frac{1-\lambda}{2}}$ attains the maximum growth and belongs to $H^{2,\lambda}$. If $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is analytic then the composition operator $C_\phi(f) = f \circ \phi$ is bounded on $H^{2,\lambda}$ and

$$\|C_\phi(f)\|_{2,\lambda} \leq C \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\frac{1-\lambda}{2}} \|f\|_{2,\lambda}.$$

More information and properties of Morrey spaces can be found in [11], [12], [20], [22], [23]

3. Semigroups in $H^{2,\lambda}$

Suppose $0 < \lambda < 1$. Since $H^\infty \subset H^{2,\lambda}$ and $H_0^{2,\lambda}$ is the closure of polynomials in $H^{2,\lambda}$, it follows from [1, Corollary 1.3] that for every semigroup (ϕ_t) ,

$$H_0^{2,\lambda} \subseteq [\phi_t, H^{2,\lambda}] \subseteq H^{2,\lambda}.$$

The question arises if there can be equality in either of the above containments. The following theorem is the analogue of Sarason's characterization of VMOA functions by their property that they can be approximated inside BMOA by their rotations or dilations.

Theorem 3.1. *Suppose $0 < \lambda < 1$. For $f \in H^{2,\lambda}$ the following are equivalent*

- (1) $f \in H_0^{2,\lambda}$.
- (2) $\lim_{t \rightarrow 0^+} \|f(e^{it}z) - f\|_{2,\lambda} = 0$.
- (3) $\lim_{t \rightarrow 0^+} \|f(e^{-t}z) - f\|_{2,\lambda} = 0$.

Proof. (1) \Rightarrow (2). Using the fact that polynomials are dense $H_0^{2,\lambda}$ and that $\|f(e^{it}z)\|_{2,\lambda} = \|f\|_{2,\lambda}$ for any $f \in H^{2,\lambda}$ and t real, a triangle inequality then shows that it suffices to check the validity of the assertion for f a polynomial, which in turn reduces to verifying it for monomials, and this is clearly true.

(2) \Rightarrow (3). Suppose (2) holds for an $f \in H^{2,\lambda}$. For $0 < r < 1$ write $f_r(z) = f(rz)$. We will show equivalently that $\lim_{r \rightarrow 1^-} \|f_r - f\|_{2,\lambda} = 0$. By the Poisson integral formula we have

$$f(e^{i\theta}) - f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t)[f(e^{i\theta}) - f(e^{i(\theta-t)})] dt$$

and an application of Fubini's theorem gives for every small positive δ ,

$$\begin{aligned} \|f - f_r\|_{2,\lambda} &\leq \frac{1}{2\pi} \int_0^{2\pi} P_r(t) \|f - f(e^{-it}z)\|_{2,\lambda} dt \\ &= \frac{1}{2\pi} \int_{|t| < \delta} P_r(t) \|f(e^{it}z) - f\|_{2,\lambda} dt \\ &\quad + \frac{1}{2\pi} \int_{\delta \leq |t| < \pi} P_r(t) \|f(e^{it}z) - f\|_{2,\lambda} dt \\ &\leq \frac{1}{2\pi} \int_{|t| < \delta} P_r(t) \|f(e^{it}z) - f\|_{2,\lambda} dt + \frac{2\|f\|_{2,\lambda}}{2\pi} \int_{\delta \leq |t| < \pi} P_r(t) dt, \end{aligned}$$

where we have again used that $\|f(e^{it}z)\|_{2,\lambda} = \|f\|_{2,\lambda}$ for $f \in H^{2,\lambda}$.

If $\varepsilon > 0$ is given, using the properties of the Poisson kernel and the assumption that (2) holds, the first integral can be made smaller than $\varepsilon/2$ by choosing δ small. Fix such a δ , then the integral in the second term tends to 0 as $r \rightarrow 1^-$, thus the second term is also less than $\varepsilon/2$ for r sufficiently close to 1. This shows that (2) implies (3).

(3) \Rightarrow (1). This implication is obvious since each f_r is analytic on a larger disc of radius $1/r$ hence $f_r \in H_0^{2,\lambda}$, and $H_0^{2,\lambda}$ is closed in $H^{2,\lambda}$. \square

The above theorem says that for $\phi_t(z) = e^{it}z$ and $\psi_t(z) = e^{-t}z$,

$$[\phi_t, H^{2,\lambda}] = [\psi_t, H^{2,\lambda}] = H_0^{2,\lambda}.$$

There are however semigroups (ϕ_t) such that the inclusion $H_0^{2,\lambda} \subset [\phi_t, H^{2,\lambda}]$ is proper. For example for $\phi_t(z) = e^{-t}z + 1 - e^{-t}$ the function

$$f_\lambda(z) = \frac{1}{(1-z)^{\frac{1-\lambda}{2}}}$$

belongs to $H^{2,\lambda} \setminus H_0^{2,\lambda}$, and satisfies

$$f_\lambda(\phi_t(z)) = e^{t\frac{1-\lambda}{2}} f_\lambda(z),$$

so

$$\|f_\lambda \circ \phi_t - f_\lambda\|_{2,\lambda} = (e^{t\frac{1-\lambda}{2}} - 1)\|f_\lambda\|_{2,\lambda} \rightarrow 0 \text{ as } t \rightarrow 0.$$

Thus $f_\lambda \in [\phi_t, H^{2,\lambda}]$ and $H_0^{2,\lambda} \subsetneq [\phi_t, H^{2,\lambda}]$. Other examples of semigroups for which $[\phi_t, H^{2,\lambda}]$ is strictly larger than $H_0^{2,\lambda}$ can be constructed easily. For example let $h(z) = (\frac{1+z}{1-z})^{\frac{1-\lambda}{2}} - 1$ and $\phi_t(z) = h^{-1}(e^{-t}h(z))$. Then

$$\|h \circ \phi_t - h\|_{2,\lambda} = (1 - e^{-t})\|h\|_{2,\lambda} \rightarrow 0 \text{ as } t \rightarrow 0,$$

so $h \in [\phi_t, H^{2,\lambda}]$ while $h \notin H_0^{2,\lambda}$.

The following theorem gives a sufficient condition on the generator G of a semigroup which implies that $[\phi_t, H^{2,\lambda}] = H_0^{2,\lambda}$.

Theorem 3.2. *Let (ϕ_t) be a semigroup of functions with generator G and $0 < \lambda < 1$. Assume that*

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2} dA(z) = 0.$$

Then $[\phi_t, H^{2,\lambda}] = H_0^{2,\lambda}$.

Proof. Since $[\phi_t, H^{2,\lambda}] = \overline{\{f \in H^{2,\lambda} : Gf' \in H^{2,\lambda}\}}$ it suffices to show that

$$\{f \in H^{2,\lambda} : Gf' \in H^{2,\lambda}\} \subset H_0^{2,\lambda}.$$

Let $g \in H^{2,\lambda}$ such that $Gg' \in H^{2,\lambda}$. We will show that $g \in H_0^{2,\lambda}$ by showing that

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|^\lambda} \int_{S(I)} |g'(z)|^2 (1-|z|) dA(z) = 0.$$

For an arc I on the circle with center $e^{i\theta}$ let $a_I = (1 - |I|)e^{i\theta} \in \mathbb{D}$. Writing $F(z) = G(z)g'(z)$ we have

$$\begin{aligned}
\frac{1}{|I|^\lambda} \int_{S(I)} |g'(z)|^2 (1 - |z|) dA(z) &= \frac{1}{|I|^\lambda} \int_{S(I)} |G(z)g'(z)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&= \frac{1}{|I|^\lambda} \int_{S(I)} |F(z)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\leq \frac{2}{|I|^\lambda} \int_{S(I)} |F(z) - F(a_I)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\quad + \frac{2}{|I|^\lambda} \int_{S(I)} |F(a_I)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\leq \frac{2}{|I|^\lambda} \int_{S(I)} |F(z) - F(a_I)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\quad + \frac{C \|F\|_{2,\lambda}^2}{(1 - |a_I|)^{1-\lambda}} \frac{2}{|I|^\lambda} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\leq \frac{2}{|I|^\lambda} \int_{S(I)} |F(z) - F(a_I)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\quad + C' \|F\|_{2,\lambda}^2 \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&= A_I + B_I,
\end{aligned}$$

where we have used the growth inequality for $F \in H^{2,\lambda}$ and the fact that $1 - |a_I| = |I|$. By the hypothesis the second term B_I tends to 0 as $|I| \rightarrow 0$. To conclude the proof it suffices to prove that $A_I \rightarrow 0$ as $|I| \rightarrow 0$.

To prove that $\lim_{|I| \rightarrow 0} A_I = 0$ recall first that for arcs $I \subset \mathbb{T}$ with $|I| < 1/2$ there is an absolute constant $C > 0$ such that

$$\frac{1 - |a_I|}{|1 - \bar{a}_I z|^2} \geq \frac{C}{|I|}$$

for all $z \in S(I)$. Next fixing for the moment an arc I , let μ be the measure on \mathbb{D} defined by

$$\mu(E) = \mu_I(E) = \int_{S(I) \cap E} \frac{1 - |z|}{|G(z)|^2} dA(z)$$

for each Borel subset E of \mathbb{D} . We will suppress the index I in μ_I until later. From the hypothesis it follows that μ is a Carleson measure, i.e. for each $f \in H^2$

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C \int_{\mathbb{T}} |f(e^{i\theta})|^2 d\theta,$$

with the constant C not depending on f , and C is comparable to

$$\|\mu\|_* = \sup_{J \subset \mathbb{T}} \frac{\mu(S(J))}{|J|}.$$

We have then

$$\begin{aligned}
A_I &= \frac{2}{|I|^\lambda} \int_{S(I)} |F(z) - F(a_I)|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&\leq C' \frac{|I|^2}{|I|^\lambda} \int_{S(I)} \left| \frac{F(z) - F(a_I)}{1 - \bar{a}_I z} \right|^2 \frac{1 - |z|}{|G(z)|^2} dA(z) \\
&= C' |I|^{2-\lambda} \int_{\mathbb{D}} \left| \frac{F(z) - F(a_I)}{1 - \bar{a}_I z} \right|^2 d\mu(z) \\
&\leq C' \|\mu\|_* |I|^{2-\lambda} \int_{\mathbb{T}} \left| \frac{F(e^{i\theta}) - F(a_I)}{1 - \bar{a}_I e^{i\theta}} \right|^2 d\theta \\
&= C' \|\mu\|_* (1 - |a_I|)^{1-\lambda} \int_{\mathbb{T}} |F(e^{i\theta}) - F(a_I)|^2 \frac{1 - |a_I|}{|1 - \bar{a}_I e^{i\theta}|^2} d\theta \\
&\leq C' \|\mu\|_* \|F\|_{2,\lambda}^2
\end{aligned}$$

where we have used the seminorm (2.7) and a change of variable in the last step.

It remains to show that $\|\mu\|_* = \|\mu_I\|_* \rightarrow 0$ as $|I| \rightarrow 0$. We have

$$\|\mu_I\|_* = \sup_{J \subset \mathbb{T}} \frac{\mu_I(S(J))}{|J|} = \sup_{J \subset \mathbb{T}} \frac{\mu_I((S(J) \cap S(I)))}{|J|}$$

so we need only consider arcs J that intersect I . Let

$$\Gamma = \{J : J \subset \mathbb{T} \text{ arc with } J \cap I \neq \emptyset\}$$

be the collection of such arcs, then

$$\Gamma = \{J \in \Gamma : |J| \geq |I|\} \cup \{J \in \Gamma : |J| < |I|\} = \Gamma_1 \cup \Gamma_2,$$

a disjoint union of two sets, and

$$\|\mu_I\|_* = \sup_{J \in \Gamma} \frac{\mu_I(S(J))}{|J|} = \sup_{J \in \Gamma_1 \cup \Gamma_2} \frac{\mu_I(S(J))}{|J|} \leq \sup_{J \in \Gamma_1} \frac{\mu_I(S(J))}{|J|} + \sup_{J \in \Gamma_2} \frac{\mu_I(S(J))}{|J|}.$$

If $J \in \Gamma_1$ then $|J| \geq |I|$ and we have

$$\frac{\mu_I(S(J))}{|J|} = \frac{\mu_I((S(J) \cap S(I)))}{|J|} \leq \frac{\mu_I(S(I))}{|J|} \leq \frac{\mu_I(S(I))}{|I|},$$

so that

$$\|\mu_I\|_* \leq \frac{\mu_I(S(I))}{|I|} + \sup_{J \in \Gamma_2} \frac{\mu_I(S(J))}{|J|}. \quad (3.1)$$

Let $\varepsilon > 0$ be given. By the hypothesis there is a $\delta > 0$ such that if $|I| < \delta$ then

$$\frac{\mu_I(S(I))}{|I|} = \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} dA(z) < \varepsilon.$$

At the same time for each $J \in \Gamma_2$ we have $|J| < |I| < \delta$ so

$$\frac{\mu_I(S(J))}{|J|} = \frac{1}{|J|} \int_{S(J) \cap S(I)} \frac{1 - |z|}{|G(z)|^2} dA(z) \leq \frac{1}{|J|} \int_{S(J)} \frac{1 - |z|}{|G(z)|^2} dA(z) < \varepsilon$$

where the last inequality is again by the hypothesis.

Thus $\sup_{J \in \Gamma_2} \frac{\mu_I(S(J))}{|J|} < \varepsilon$, and it follows from (3.1) that $\|\mu_I\|_* \leq 2\varepsilon$ for $|I| < \delta$, which finishes the proof. \square

As a corollary we obtain an analogue of [4, Theorem 3.1].

Corollary 3.3. *Let (ϕ_t) be a semigroup of functions with generator G and $0 < \lambda < 1$. Assume that for some α with $0 < \alpha < 1$,*

$$\frac{(1 - |z|)^\alpha}{G(z)} = O(1), \quad \text{as } |z| \rightarrow 1.$$

Then $[\phi_t, H^{2,\lambda}] = H_0^{2,\lambda}$.

Proof. Under the hypothesis, there is some $r_0 < 1$ such that $\frac{(1-|z|)^{2\alpha}}{|G(z)|^2} \leq C < \infty$ for $r_0 < |z| < 1$. Then

$$\frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} dA(z) \leq C \frac{1}{|I|} \int_{S(I)} (1 - |z|)^{1-2\alpha} dA(z) \leq C'|I|^{2-2\alpha}$$

so that $\lim_{|I| \rightarrow 0} \frac{1}{|I|} \int_{S(I)} \frac{1-|z|}{|G(z)|^2} dA(z) = 0$ and the conclusion follows from the previous theorem. \square

The following is a partial converse of the above corollary for semigroups with Denjoy-Wolff point inside the disc.

Theorem 3.4. *Let (ϕ_t) be a semigroup with infinitesimal generator G and Denjoy-Wolff $b \in \mathbb{D}$. If for some $\lambda \in (0, 1)$ we have $[\phi_t, H^{2,\lambda}] = H_0^{2,\lambda}$, then*

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

Proof. Without loss of generality we may assume $b = 0$. The generator then is $G(z) = -zP(z)$ where $\operatorname{Re}(P) \geq 0$ on \mathbb{D} . We assume that P is nonconstant (for P constant the assertion is clear). Let

$$\psi(z) = \int_0^z \frac{\zeta}{G(\zeta)} d\zeta = - \int_0^z \frac{1}{P(\zeta)} d\zeta.$$

Since $\operatorname{Re}(1/P) \geq 0$, the Taylor coefficients of $1/P$ is a bounded sequence, so the Taylor coefficients $\widehat{\psi}(n)$ of the function $\psi(z)$ satisfy $|\widehat{\psi}(n)| \leq C/n$ for some positive constant C . It follows [10, Theorem 9.4] that ψ belongs to $BMOA$ and thus also to $H^{2,\lambda}$. In addition,

$$G(z)\psi'(z) = z,$$

a polynomial of degree 1 which belongs to $H^{2,\lambda}$. Since both ψ and $G\psi'$ belong to $H^{2,\lambda}$ it follows from (2.2) that $\psi \in [\phi_t, H^{2,\lambda}]$ and by the hypothesis $\psi \in H_0^{2,\lambda}$.

Now let $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ with $a \in \mathbb{D}$. Then $(\psi \circ \phi_a)'(0) = \psi'(a)(|a|^2 - 1)$ and we have

$$\begin{aligned} \frac{|a|(1-|a|^2)^{\frac{3-\lambda}{2}}}{|G(a)|} &= (1-|a|^2)^{\frac{3-\lambda}{2}} |\psi'(a)| = (1-|a|^2)^{\frac{1-\lambda}{2}} |(\psi \circ \phi_a)'(0)| \\ &\leq (1-|a|^2)^{\frac{1-\lambda}{2}} \|\psi \circ \phi_a\|_{H^2} \\ &\leq (1-|a|^2)^{\frac{1-\lambda}{2}} |\psi(a)| + (1-|a|^2)^{\frac{1-\lambda}{2}} \|\psi \circ \phi_a - \psi(a)\|_{H^2}. \end{aligned}$$

But since $\psi \in H_0^{2,\lambda}$, we have $\lim_{|a| \rightarrow 1} (1-|a|^2)^{\frac{1-\lambda}{2}} \|\psi \circ \phi_a - \psi(a)\|_{H^2} = 0$ and also $\lim_{|a| \rightarrow 1} (1-|a|^2)^{\frac{1-\lambda}{2}} |\psi(a)| = 0$, so the conclusion follows. \square

3.1. Strong continuity on the whole space

As we have mentioned earlier, for every nontrivial semigroup (ϕ_t) the maximal space of strong continuity $[\phi_t, X]$ is a proper subspace of X when $X = H^\infty$ or $X = \mathcal{B}$, the Bloch space. A proof of this uses the fact that each space is a Grothendieck space and has the Dunford-Pettis property. H. Lotz [13] has proved that if X is a Banach space with these two properties then every strongly continuous operator semigroup is automatically continuous in the uniform operator topology of X . This means that the infinitesimal generator is a bounded operator on X . In the case of composition semigroups the infinitesimal generator is a differential operator, and as such it is not bounded on H^∞ or on \mathcal{B} unless it is the zero operator. The same phenomenon appears on all generalized Bloch spaces \mathcal{B}^α , $\alpha > 0$, which are defined by

$$\mathcal{B}^\alpha = \{f : \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\},$$

with norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |f'(z)|$. Each such space is a Grothendieck spaces with the Dunford-Pettis property see [5], so the theorem of Lotz applies.

The space BMOA does not have the geometric Banach space properties mentioned above. Nevertheless $[\phi_t, BMOA]$ is also a proper subspace of BMOA for every nontrivial (ϕ_t) . This was proved in [1] in a more general form:

Theorem([1, Theorem 1.1]) *Let X be a Banach space of analytic functions on \mathbb{D} such that $H^\infty \subseteq X \subseteq \mathcal{B}$. Then for every nontrivial (ϕ_t) , $[\phi_t, X] \subsetneq X$.*

Note that this theorem gives a new function theoretic proof for H^∞ and \mathcal{B} . The proof consists of finding a suitable test function f in H^∞ (interpolating Blaschke product with zeros along a radius) such that $\limsup_{r \rightarrow 1^-} |f'(r)|(1-r) > 0$ and f extends to be analytic at a neighborhood of all boundary points $\zeta \in \mathbb{T} \setminus \{1\}$. This then is used, together with the boundary behavior of the associated univalent function of (ϕ_t) , analyzed with the aid of the theory of prime ends, to show that the Bloch norm of $f \circ \phi_t - f$ stays away from zero as t approaches 0. These arguments can be adapted to the case of Morrey spaces, with different test function, to prove an analogous result.

Theorem 3.5. *Let $\lambda \in (0, 1)$ and let X be a Banach space of analytic functions on \mathbb{D} such that $H^{2,\lambda} \subseteq X \subseteq \mathcal{B}^{\frac{3-\lambda}{2}}$. Then for every nontrivial semigroup (ϕ_t) of functions we have $[\phi_t, X] \subsetneq X$. In particular $[\phi_t, H^{2,\lambda}] \subsetneq H^{2,\lambda}$.*

Proof. The proof is based on the following claim (see also Theorem 3.1 in [1]).

Claim For every nontrivial (ϕ_t) there is a function $f \in H^{2,\lambda}$ such that

$$\liminf_{t \rightarrow 0} \|f \circ \phi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq 1.$$

Proof of the Claim. Without loss of generality we may assume that the Denjoy-Wolff point b is either 0 or 1. Take the case $b = 0$ first. Then the semigroup is

$$\phi_t(z) = h^{-1}(e^{-ct}h(z))$$

where h is a univalent spirallike function mapping \mathbb{D} onto $\Omega = h(\mathbb{D})$ with $h(0) = 0$ and $\operatorname{Re}(c) \geq 0$.

If $\operatorname{Re}(c) = 0$ then $\phi_t(z)$ are rotations and have the form $\phi_t(z) = e^{iat}z$ with $a \in \mathbb{R} - \{0\}$. Let

$$f(z) = \frac{1}{(1-z)^{\frac{1-\lambda}{2}}}$$

then $f \in H^{2,\lambda}$ and

$$\lim_{r \rightarrow 1^-} |f'(re^{i\theta})|(1-r)^{\frac{3-\lambda}{2}} = \begin{cases} \frac{1-\lambda}{2} > 0, & \text{when } \theta = 0, \\ 0, & \text{when } 0 < \theta < 2\pi. \end{cases}$$

Now for $0 < t < 2\pi/|a|$ we have

$$\begin{aligned} \|f \circ \phi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} &\geq \sup_{0 < r < 1} |f'(re^{iat})e^{iat} - f'(r)|(1-r)^{\frac{3-\lambda}{2}} \\ &\geq \limsup_{r \rightarrow 1^-} |f'(re^{iat})e^{iat} - f'(r)|(1-r)^{\frac{3-\lambda}{2}} \\ &\geq \frac{1-\lambda}{2}. \end{aligned}$$

Consider now the case $\operatorname{Re}(c) > 0$. As in Theorem 3.1 of [1] we can choose a point $w \in \partial\Omega$ such that

$$|w| = \operatorname{dist}(0, \partial\Omega).$$

Then $[0, w) \subset \Omega$ and we can view the segment $[0, w)$ as a curve $\gamma(s) = sw$, $0 \leq s < 1$, ending at the point $w \in \partial\Omega$. Then the inverse image $\Gamma(s) = h^{-1}(\gamma(s))$ is a curve in \mathbb{D} ending at some point $\zeta \in \mathbb{T}$ [15, Proposition 2.14]. Thus h has the limit w along the curve Γ and by [15, Corollary 2.17] it has radial limit w at ζ , i.e. $\lim_{r \rightarrow 1^-} h(r\zeta) = w$. For each $t > 0$ then,

$$\lim_{r \rightarrow 1^-} \phi_t(r\zeta) = \lim_{r \rightarrow 1^-} h^{-1}(e^{-ct}h(r\zeta)) = h^{-1}(e^{-ct}w) \in \mathbb{D}.$$

Since ϕ_t is a bounded univalent function, $\lim_{r \rightarrow 1^-} |\phi'_t(r\zeta)|(1-r) = 0$ so also $\lim_{r \rightarrow 1^-} |\phi'_t(r\zeta)|(1-r)^{\frac{3-\lambda}{2}} = 0$. Letting $f_\zeta(z) = f(\bar{\zeta}z)$ we have

$$\lim_{r \rightarrow 1^-} |f'_\zeta(r\zeta)|(1-r)^{\frac{3-\lambda}{2}} = \lim_{r \rightarrow 1^-} |f'(r)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2}.$$

In addition f'_ζ is continuous on \mathbb{D} so for fixed $t > 0$,

$$\lim_{r \rightarrow 1^-} |f'_\zeta(\phi_t(r\zeta))| = |f'_\zeta(h^{-1}(e^{-ct}w))| < \infty.$$

Thus for all $t > 0$,

$$\|f_\zeta \circ \phi_t - f_\zeta\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \limsup_{r \rightarrow 1^-} |f'_\zeta(\phi_t(r\zeta))\phi'_t(r\zeta) - f'_\zeta(r\zeta)|(1-r)^{\frac{3-\lambda}{2}} \geq \frac{1-\lambda}{2}. \quad (3.2)$$

Thus in both cases $\|f \circ \phi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \frac{1-\lambda}{2} > 0$ for all t close to 0.

Multiplying the test function in each case by $2/(1-\lambda)$ gives the result.

If the Denjoy-Wolff point b is the point 1 then

$$\phi_t(z) = h^{-1}(h(z) + ct)$$

where h is a univalent function mapping \mathbb{D} onto a close-to-convex domain Ω with $h(0) = 0$ and $\operatorname{Re}(c) > 0$. If (ϕ_t) consists of automorphisms of the disc then the map $z \rightarrow z + ct$ is an automorphism of Ω for each t , so Ω is either a half-plane or a strip, and the finite part of $\partial\Omega$ is either a line or two parallel lines. Take any point w in the finite part of $\partial\Omega$, then there is a point $\zeta \in \mathbb{T}$ such that $h(\zeta) = w$. Then $\phi_t(\zeta) \in \partial\mathbb{D}$ and for each $t > 0$, $\phi_t(\zeta) \neq \zeta$. Let $f_\zeta(z) = (1 - \bar{\zeta}z)^{-\frac{1-\lambda}{2}}$. Then for $\beta \in \mathbb{T}$,

$$\lim_{r \rightarrow 1^-} |f'_\zeta(r\beta)|(1-r)^{\frac{3-\lambda}{2}} = \begin{cases} \frac{1-\lambda}{2}, & \text{when } \beta = \zeta, \\ 0, & \text{when } \beta \neq \zeta. \end{cases}$$

In addition f'_ζ extends continuously on $\overline{\mathbb{D}} \setminus \{\zeta\}$, and in particular at each point $\zeta_t = \phi_t(\zeta) \in \mathbb{T}$, $t > 0$. Note that $\zeta_t \neq \zeta$. Thus

$$\lim_{r \rightarrow 1^-} f'_\zeta(\phi_t(r\zeta)) = f'_\zeta(\zeta_t).$$

Also ϕ'_t is a bounded function on \mathbb{D} for each $t > 0$. For fixed t then

$$\|f_\zeta \circ \phi_t - f_\zeta\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \limsup_{r \rightarrow 1^-} |f'_\zeta(\phi_t(r\zeta))\phi'_t(r\zeta) - f'_\zeta(r\zeta)|(1-r)^{\frac{3-\lambda}{2}} \geq \frac{1-\lambda}{2},$$

and replacing f_ζ by $\frac{2}{1-\lambda}f_\zeta$ we have the assertion in the case when ϕ_t are automorphisms.

Finally if ϕ_t are not automorphisms, the map $z \rightarrow z + ct$ is not onto Ω . Following the arguments of the analogous case in Theorem 3.1 of [1], let $t > 0$ and pick a $z \in \Omega \setminus (\Omega + ct)$. Then there is $w \in \partial\Omega$ and $t_0 \in (0, t]$ such that $w + ct_0 = z$ and $(w, z] \subset \Omega$. There is a $\zeta \in \mathbb{T}$ such that $\lim_{r \rightarrow 1^-} h(r\zeta) = w$, and the argument for the case of interior Denjoy-Wolff point repeats word-for-word to obtain a positive lower bound for the norm $\|f_\zeta \circ \phi_t - f_\zeta\|_{\mathcal{B}^{\frac{3-\lambda}{2}}}$ as in (3.2). Again replacing f_ζ by $\frac{2}{1-\lambda}f_\zeta$ gives the assertion of the claim.

Having proved the claim we continue to finish the proof of the theorem. The test functions we used above are in $H^{2,\lambda}$ so also in X . The identity embedding map $i : X \rightarrow \mathcal{B}^{\frac{3-\lambda}{2}}$ has closed graph so by the Closed Graph Theorem it is a bounded operator, which means that there is a constant C

such that $\|f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \leq C\|f\|_X$ for each $f \in X$. In particular then the test functions that we have used are not in $[\phi_t, X]$ so $[\phi_t, X] \subsetneq X$. \square

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