Awareness of and *awareness that*: their combination and dynamics

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Abstract

The paper proposes a logical framework representing the notion of *explicit knowledge* as the combination of *awareness of* and *awareness that*. The setting, semantically combining neighbourhood models with ideas from awareness logic, separates the mere fact of entertaining some information (being *aware of* φ) from the acknowledgement that the information is indeed the case (being *aware that* φ holds). The text discusses not only the main properties these concepts obtain under the given representation, but also several of the epistemic actions that can be defined, and the way they affect the agent's awareness (and thus her knowledge).

Keywords: Awareness, explicit knowledge, epistemic logic, awareness logic, neighbourhood semantics, epistemic actions, observation, deduction, awareness change, dynamic epistemic logic.

1 Introduction

The different solutions to the problem of logical omniscience [35, 57] propose frameworks for modelling the knowledge of *'real' agents* with limited reasoning abilities. One successful strategy has been to split knowledge into *explicit* (what a 'real' agent has) and *implicit* (what an ideal agent with unlimited resources would obtain). A prosperous proposal that followed this strategy is *awareness logic* [24]: for a 'real' agent to know φ explicitly, φ not only needs to be part of her epistemic alternatives (as in standard epistemic logic), she also needs to be *aware of* φ. Applications in Philosophy, Computer Science and Economics (see, e.g. the comprehensive handbook chapter [50]) have proven this idea to be very useful.

Nevertheless, the concept of 'awareness' is open for different interpretations (cf. [21]); the agent can be *aware of* φ, i.e. she might entertain φ without having any attitude in favour or against it, or she can be *aware that* φ is the case, meaning that she acknowledges φ's truth. Thus, the potential *lack* of awareness is twofold: lacking *awareness of* renders agents who, though not aware of all possibilities, remain ideal (omniscient) regarding what they entertain (see, e.g. the original [24], and also [33, 34]); lacking *awareness that* yields agents who, while entertaining all relevant possibilities, may not be able to realize that a certain φ is the case, despite having explicitly enough information to deduce it (see, e.g. [40, 59]).

A previous work [26] proposed a theoretical framework[1] where explicit knowledge is given by the combination of *awareness of* and *awareness that*, separating thus the mere fact of entertaining some information (being *aware of* φ; a matter of attention) from acknowledging that the information is

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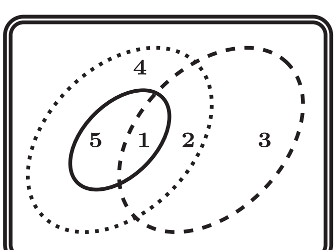
[1] Cf. the proposal in [32].

Vol. 00, No. 0,
doi:10.1093/jigpal/jzz043

indeed the case (being *aware that* φ holds; having some sort of 'evidence' for it). The diagram on Figure 1 shows this setting, where the combination of *awareness of* and *awareness that* delivers further epistemic concepts. While the large dashed ellipse on the right contains what the agent is aware of, the small solid ellipse near the centre contains what the agent is aware that. Two further 'big areas' arise: the logical consequences of what the agent is aware that (the large dotted ellipse on the left), and all truthful information the agent might become aware of (the whole domain). The first can be seen as the agent's implicit knowledge under acts of deductive inference (what she would recognize as true after performing all possible deductive inferences); the second can be seen as the agent's implicit information under acts of becoming aware (what she would entertain if she became aware of all relevant possibilities). The regions the ellipses define are described by the text next to the diagram.

This manuscript uses the notions of *awareness of* and *awareness that* to provide a logical framework that models the knowledge of agents in a more realistic way. The main ideas behind the proposal are the following. First, a 'real' agent might not entertain all relevant possibilities (she might lack *awareness of*), and she might not acknowledge as true the logical consequences of what she has already accepted (she might lack *awareness that*). Second, considering these two different forms of awareness *together* allows a richer (and thus more realistic) representation of epistemic states. For example, it is possible to have an agent that, while trying to prove a mathematical theorem, has accepted the hypotheses as true and is aware of the conclusion, and still fails to accept the latter (yet). Finally, even though an agent does not need to be aware of all possibilities and does not need to acknowledge all logical consequences of what she already has, this does not mean that she is logically ignorant; she might become aware of new alternatives, and she might perform inferences to find out that some additional facts are indeed the case. In other words, a 'logically competent' agent does not have to be logically omniscient; it is enough for her to be able to perform different *epistemic actions* to change (and, sometimes, improve) her information.

The proposal starts (Section 2) by providing a formal framework in which both notions of awareness are represented. It continues (Section 3) with a discussion about the properties these and other derived epistemic concepts (crucially, that of *explicit knowledge*) obtain under such representation, as well as a brief comparison with related alternatives. The last part (Section 4) is devoted to the representation of diverse epistemic actions, and the way they affect the agent's information. Section 5 summarizes the proposal, listing also relevant ongoing and future work.



1 - What the agent is aware that and aware of (i.e., what she explicitly knows).
2 - What the agent is entertaining, has not recognised as true, but will do after deductive reasoning. (i.e., what she implicitly knows).
3 - What the agent is entertaining, has not recognised as true, and is outside the scope of deductive reasoning.
4 - What the agent could deduce if she became aware of all her information.
5 - What the agent has recognised as true but is not currently entertaining.

FIGURE 1. Combining *awareness of* and *awareness that*. The dashed ellipse contains what the agent is *aware of*, the solid ellipse contains what the agent is *aware that* and the dotted ellipse contains the *logical consequences of awareness that*.

2 Basic framework

This section introduces the basic framework[2] for depicting explicit knowledge as the combination of *awareness of* and *awareness that*. On the semantic side, the structure used for modelling such concepts is a neighbourhood model ([43, 51]; see [14, Chapter 7] and [46] for a modern presentation) extended with a global atomic awareness set (cf. [24]). The neighbourhood function is used for representing what the agent is *aware that*, with each world's neighbourhood understood as a list containing the semantic representation of the formulas the agent has acknowledged as true. This gives us a notion of *awareness that* that is not closed under logical consequence, as one gets when using more standard relational 'Kripke' models. The atomic awareness set is used for representing what the agent is *aware of*. This gives us a notion of *awareness of* that can be understood as the atoms defining the agent's current *language*.

In this text, let $\mathtt{P}$ be a non-empty enumerable set of atomic propositions.

DEFINITION 1 (Awareness neighbourhood model).
An *awareness neighbourhood* model (*ANM*) is a tuple $M = \langle W, N, V, A\rangle$ where (i) W, also denoted as $\mathcal{D}_M$, is a non-empty set (of *possible worlds*); (ii) $N : W \to \wp(\wp(W))$ is a *neighbourhood function* (assigning a set of sets of worlds to each possible world, with $N(w)$ called the *neighbourhood* of w); (iii) $V : \mathtt{P} \to \wp(W)$ is an *atomic valuation function* (indicating the set of possible worlds in which each atom is true); (iv) $A \subseteq \mathtt{P}$ is the *atomic awareness set* (indicating the set of atoms the agent is aware of).

On the syntactic side, the language extends the propositional one with operators for describing the agent's *awareness of* (A^{o}), her *awareness that* (A^{t}) and the deductive closure of the latter ($[*]$).

DEFINITION 2 (Language and semantic interpretation).
Formulas φ, ψ of the language $\mathcal{L}$ are given by

$$\varphi, \psi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathrm{A}^{\mathrm{o}}\,\varphi \mid \mathrm{A}^{\mathrm{t}}\,\varphi \mid [*]\,\varphi$$

with $p \in \mathtt{P}$. Formulas of the form $\mathrm{A}^{\mathrm{o}}\,\varphi$ are read as *'the agent is aware of φ'*, those of the form $\mathrm{A}^{\mathrm{t}}\,\varphi$ are read as *'the agent is aware that φ'* and $[*]\,\varphi$ expresses that *'after the agent performs every possible deductive inference, φ holds'*. For any formula $\varphi \in \mathcal{L}$, define its set of atoms, $\mathrm{at}(\varphi)$, in the standard way:

$$\begin{array}{ccc} \mathrm{at}(\top) := \varnothing, & \mathrm{at}(\neg\varphi) := \mathrm{at}(\varphi), & \mathrm{at}(\mathrm{A}^{\mathrm{o}}\,\varphi) := \mathrm{at}(\varphi), \\ \mathrm{at}(p) := \{p\}, & \mathrm{at}(\varphi \wedge \psi) := \mathrm{at}(\varphi) \cup \mathrm{at}(\psi), & \mathrm{at}(\mathrm{A}^{\mathrm{t}}\,\varphi) := \mathrm{at}(\varphi), \\ & & \mathrm{at}([*]\,\varphi) := \mathrm{at}(\varphi). \end{array}$$

Given an *ANM* $M = \langle W, N, V, A\rangle$ and a formula $\varphi \in \mathcal{L}$, the function $[\![\cdot]\!]^M : \mathcal{L} \to \wp(W)$ returns the set of worlds in M at which φ holds ($[\![\varphi]\!]^M$ is φ's *truth set*). In its (inductive) definition, the cases for $\top$, atoms and Boolean operators are standard:

$$[\![\top]\!]^M := W, \quad [\![p]\!]^M := V(p), \quad [\![\neg\varphi]\!]^M := W \setminus [\![\varphi]\!]^M, \quad [\![\varphi \wedge \psi]\!]^M := [\![\varphi]\!]^M \cap [\![\psi]\!]^M.$$

For $\mathrm{A}^{\mathrm{o}}\,\varphi$ (awareness of), the formula is either globally true (when the agent is aware of all φ's atoms) or else globally false (otherwise):

$$[\![\mathrm{A}^{\mathrm{o}}\,\varphi]\!]^M := \begin{cases} W & \text{if } \mathrm{at}(\varphi) \subseteq A; \\ \varnothing & \text{otherwise.} \end{cases}$$

[2] A previous work, containing a preliminary version of this framework, can be found in [26].

For $\mathrm{A}^{\mathrm{t}}\,\varphi$ (awareness that), the formula is true at a world w in M if and only if the truth set of φ in M is in the neighbourhood of w:

$$[\![\mathrm{A}^{\mathrm{t}}\,\varphi]\!]^{M} := \left\{ w \in W \mid [\![\varphi]\!]^{M} \in N(w) \right\}.$$

The remaining modality, $[*]$, is a special case, as its semantic interpretation relies not only on the given M, but also on its *augmentation*: the model M^{*} that results from making the neighbourhood of each world a set that contains the neighbourhood's *core*, $\bigcap N(w)$, and is closed under supersets.[3] [4] More precisely, from the given $M = \langle W, N, V, A \rangle$, define $M^{*} = \langle W, N^{*}, V, A \rangle$ with N^{*} such that

$$N^{*}(w) := \left\{ U \subseteq W \mid \bigcap N(w) \subseteq U \right\}.$$

Then,

$$[\![[*]\,\varphi]\!]^{M} := [\![\varphi]\!]^{M^{*}}.$$

As it will be recalled (Section 3.2), in M^{*} the modality A^{t} behaves as $\Box$ does in relational models; this is the reason behind intuitive reading of formulas of the form $[*]\,\varphi$, and the reason why here $[*]$ is also called the *deductive closure* modality.

The concepts of *satisfiability* and *validity* are defined in the standard way, with the latter denoted as usual ($\Vdash \varphi$).

Thus, while *awareness that* (A^{t}) corresponds to the (local) neighbourhood function N, *awareness of* (A^{o}) is generated by the (global) set of atoms A.[5]

3 Concepts, their properties and their relationship

The just introduced formal framework allows us to provide formal definitions for the notions the diagram on section 1 sketches. Besides *awareness of* and *awareness that*, the concepts that will be discussed in detail are the following.

- **Explicit knowledge** is what the agent is entertaining (i.e. is aware of) and has also acknowledged as true (i.e. is aware that):

  $$\mathrm{K}_{Ex}\,\varphi := \mathrm{A}^{\mathrm{o}}\,\varphi \wedge \mathrm{A}^{\mathrm{t}}\,\varphi.$$

  In this sense, explicit knowledge can be understood as what the agent 'really' knows at the given stage.
- **Implicit knowledge** is what the agent is entertaining (i.e. is aware of) and will acknowledge as true after applying all possible deductive inferences:

  $$\mathrm{K}_{Im}\,\varphi := \mathrm{A}^{\mathrm{o}}\,\varphi \wedge [*]\,\mathrm{A}^{\mathrm{t}}\,\varphi.$$

  In other words, implicit knowledge is what the agent can deduce from what she knows explicitly.

Thus, while explicit knowledge is what the agent has at the given stage, implicit knowledge is all she will have after full deductive closure.

[3] For more details on this model construction, see [14, Section 7.3].

[4] This is the *dynamic epistemic logic* approach (*DEL*; [7, 18]), defining operations that change the underlying model, and then using modalities that are interpreted over the models that result from such operations.

[5] For space reasons, the discussion on the axiom system characterizing the formulas that are valid over *ANM*s is left for a companion manuscript.

Explicit and implicit knowledge require *awareness of*, as both deal with the possibilities the agent is currently entertaining. But there are similar notions of knowledge, having the same relationship with each other (one the closure under deductive inference of the other), but staying outside of the possibilities the agent is currently pondering. The first one, called *'disassociated' knowledge*, is what the agent has acknowledged as true but is not currently entertaining (that is, what will become explicitly known after she becomes aware of it), and is given by $\mathrm{K}_{Ex}^{-o}\,\varphi := \neg\,\mathrm{A^o}\,\varphi \wedge \mathrm{A^t}\,\varphi$. The second one, called *currently 'unreachable' knowledge*, is what the agent is not currently entertaining, and yet she can deduce from what she has acknowledged as true (i.e. what she could deduce after becoming aware of what she is aware that), and is given by $\mathrm{K}_{Im}^{-o}\,\varphi := \neg\,\mathrm{A^o}\,\varphi \wedge [*]\,\mathrm{A^t}\,\varphi$.

The rest of this section discusses the properties some of these epistemic concepts obtain under the given semantic model and definitions. The focus will be the two primitive concepts, *awareness of* and *awareness that*, as well as *explicit knowledge* and its *implicit* counterpart.

3.1 Basic properties and relationships

***Awareness of*.** This concept, $\mathrm{A^o}$, is understood here as what the agent *entertains*. Thus, *awareness of* is a matter of attention, and by itself it does not imply any attitude pro or con: the agent may have accepted the formula as true, as false, or she might not have any inclination about it.

Semantically, $\mathrm{A^o}$ is defined in terms of a global set of atomic propositions, the atomic awareness set A. More precisely, given an *ANM* model $M = \langle W, N, V, A\rangle$, the agent is aware of a given formula φ at a world $w \in W$ if and only if all atoms occurring in φ belong to this set, $\mathrm{at}(\varphi) \subseteq A$. As a consequence of this, the agent is aware of the concept of 'truth':

$$\Vdash \mathrm{A^o}\,\top.$$

This does not say that the agent is aware that this 'truth' holds everywhere; it simply states that she entertains such concept. Technically, the reason is that $\top$, a primitive in the language (thus not defined as an abbreviation of the form $p \vee \neg p$, as in other proposals) does not contain any atomic proposition.

Still, despite being aware of the concept of truth, the agent does not need to be aware of formulas that are true in every possible situation (that is, she does not need to be aware of valid formulas):

- $\Vdash \varphi$ does not imply $\Vdash \mathrm{A^o}\,\varphi$.

Technically, this is the case because a validity might have atoms (e.g. $p \vee \neg p$), and no atom is required to be in the atomic awareness set (e.g. p does not need to be in A); hence, there might be validities the agent does not entertain. Along the same lines, *awareness of* is not closed under logical equivalence, as the given formulas might involve different atoms:

- $\Vdash \varphi \leftrightarrow \psi$ does not imply $\Vdash \mathrm{A^o}\,\varphi \leftrightarrow \mathrm{A^o}\,\psi$.

Note that *awareness of* is defined not in terms of a set of formulas (as in the logic of general awareness of [24]) but rather in terms of a set of *atomic propositions*. As discussed already in the mentioned paper, this makes the concept of *awareness of* closed not only under subformulas but also under superformulas. More precisely,

$$\Vdash \mathrm{A^o}\,\neg\varphi \leftrightarrow \mathrm{A^o}\,\varphi, \qquad \Vdash \mathrm{A^o}\,\mathrm{A^o}\,\varphi \leftrightarrow \mathrm{A^o}\,\varphi,$$
$$\Vdash \mathrm{A^o}(\varphi \wedge \psi) \leftrightarrow (\mathrm{A^o}\,\varphi \wedge \mathrm{A^o}\,\psi), \qquad \Vdash \mathrm{A^o}\,\mathrm{A^t}\,\varphi \leftrightarrow \mathrm{A^o}\,\varphi,$$
$$\Vdash \mathrm{A^o}\,[*]\,\varphi \leftrightarrow \mathrm{A^o}\,\varphi.$$

In particular, note what the formulas on the right column indicate. The first states that awareness of awareness of a formula is equivalent to awareness of the formula. The other two indicate, with some paraphrasing (and read from right to left), that entertaining φ is equivalent to entertaining the possibility of having accepted φ as tr ue (the second) and also equivalent to entertaining the possibility of obtaining φ after deductive inference (the third).

An important difference between the *awareness of* discussed here and the original one in [24] is that, while the latter is local (what belongs to the awareness set assigned to the given evaluation point), the one proposed here is global (what belongs to the atomic awareness set assigned to the whole model). An intermediate position is held in the relational-model-based approach of [32], where (atom-based) *awareness of* is defined as those formulas whose atoms appear in the awareness set of all worlds the agent cannot distinguish from the evaluation point. The definition makes sense in the multi-agent setting that the referred paper studies; in the single-agent case examined here, one can assume that the worlds in the model are exactly those that are relevant for the agent under discussion, thus making the two definitions (that of [32] and the one proposed here) conceptually equivalent.

Awareness that. The concept of *awareness that*, $\mathrm{A^t}$, is understood here as what the agent has accepted/acknowledged as true; in this sense, it is a form of 'explicit' information. Still, it is not called *explicit knowledge* because the agent might not entertain such piece of information at the current stage. Thus, even though *awareness that* is acknowledgement of truth, such acceptance does not imply by itself that the agent is entertaining such piece of information (she might have accepted it before, and then moved on to a different topic), and therefore it does not imply explicit knowledge either.

Semantically, $\mathrm{A^t}$ is defined as what appears in the neighbourhood of the current evaluation point.[6] In fact, the neighbourhood of a given world, a set of sets of worlds, can be understood as the list of formulas the agent has acknowledged as true, the important point being that these formulas are not represented syntactically (as a string of symbols), but rather semantically (as the set of those worlds in the model in which the formula holds). Because of this purely semantic representation, the concept of *awareness that* has an important closure property; it is closed under logical equivalence:

$$\Vdash \varphi \leftrightarrow \psi \quad \text{implies} \quad \Vdash \mathrm{A^t}\,\varphi \leftrightarrow \mathrm{A^t}\,\psi.$$

Thus, the agent has indeed some omniscience; at the level of acknowledgement, she cannot tell apart formulas that are true in exactly the same worlds in all models.[7] Still, as it will be discussed later, this does not mean that the agent's *explicit knowledge* is closed under logical equivalence; the concept of *awareness of* will help to distinguish between logically equivalent formulas.

Closure under logical equivalence is the only closure property the notion of *awareness that* has. Different from other semantic representations of information, as the $\Box$ operator in standard epistemic logic under relational models, (i) validities do not need to be acknowledged as true, (ii) *awareness that* is not closed under conjunction introduction and (iii) neither under conjunction elimination .

(i) $\Vdash \varphi$ does not imply $\Vdash \mathrm{A^t}\,\varphi$,
(ii) $\nVdash (\mathrm{A^t}\,\varphi \wedge \mathrm{A^t}\,\psi) \rightarrow \mathrm{A^t}(\varphi \wedge \psi)$,
(iii) $\nVdash \mathrm{A^t}(\varphi \wedge \psi) \rightarrow \mathrm{A^t}\,\varphi$ and $\nVdash \mathrm{A^t}(\varphi \wedge \psi) \rightarrow \mathrm{A^t}\,\psi$.

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[6] As readers familiar with neighbourhood semantics might have noticed, $\mathrm{A^t}$ uses the 'set' semantic interpretation of the $\Box$ operator in neighbourhood models. (Recall: the 'subset' semantic interpretation makes $\Box\varphi$ true at w in M not only when $[\![\varphi]\!]^M$ is in $N(w)$, but also when any of its *subsets* is $[\![\Box\varphi]\!]^M := \{w \in W \mid \text{there is } U \in N(w) \text{ such that } U \subseteq [\![\varphi]\!]^M\}$.)

[7] This concept of *awareness that*, sometimes called *explicit knowledge*, does not have this closure property in other proposals (e.g. [32, 40]), simply because it is represented by a set of formulas that is not required to have any closure property.

The reason for the failure of these properties is that no neighbourhood needs to have any closure property. In particular, (i) none of them needs to contain the whole domain (so $\mathcal{D}_M$, the truth set of any validity, does not need to be in $N(w)$), (ii) none of them needs to be closed under intersections (so $[\![\varphi]\!]^M, [\![\psi]\!]^M \in N(w)$ does not imply $[\![\varphi]\!]^M \cap [\![\psi]\!]^M = [\![\varphi \land \psi]\!]^M \in N(w)$) and (iii) none of them needs to be closed under supersets (so $[\![\varphi \land \psi]\!]^M = [\![\varphi]\!]^M \cap [\![\psi]\!]^M \in N(w)$ implies neither $[\![\varphi]\!]^M \in N(w)$ nor $[\![\psi]\!]^M \in N(w)$) . Because of the failure of conjunction introduction/elimination, *awareness that* is not closed under modus ponens:[8]

- $\not\Vdash \mathrm{A}^{\mathrm{t}}(\varphi \to \psi) \to (\mathrm{A}^{\mathrm{t}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\psi)$.

Hence, *awareness that* is not closed under logical consequence.

***Awareness of* and *awareness that*.** A property relating A^{o} and A^{t} has been discussed already ($\Vdash \mathrm{A}^{\mathrm{o}}\,\mathrm{A}^{\mathrm{t}}\,\varphi \leftrightarrow \mathrm{A}^{\mathrm{o}}\,\varphi$). Yet, for readers familiar with awareness logic, the fact that *awareness of* is global might suggest a further relationship: that the agent 'knows her own awareness'. Indeed, in the original [24], the fact that the awareness set of all worlds in the model is the same implies not only $\Vdash \mathrm{A}\,\varphi \to \Box\,\mathrm{A}\,\varphi$ (if the agent is aware of φ, then she knows this [implicitly]) but also $\Vdash \neg\,\mathrm{A}\,\varphi \to \Box\,\neg\,\mathrm{A}\,\varphi$ (if she is not aware of φ, then she knows this [implicitly]).

In the present setting, analogous properties do not need to hold. The agent does not need to acknowledge any formula, even if it holds in every world of the model; thus, she does not need to acknowledge her *awareness* and neither her *unawareness*.

- $\not\Vdash \mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{o}}\,\varphi$,
- $\not\Vdash \neg\,\mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{o}}\,\varphi$.

Explicit knowledge. This concept, K_{Ex}, is defined as those pieces of information the agent has acknowledged as true and is currently entertaining, i.e. as what the agent is both *aware of* and *aware that*: $\mathrm{K}_{Ex}\,\varphi := \mathrm{A}^{\mathrm{o}}\,\varphi \land \mathrm{A}^{\mathrm{t}}\,\varphi$.

About explicit knowledge of validities. Since the agent needs to be neither aware of them nor aware that they are the case, she does not need to know explicitly any validity. Moreover, none of the awareness concepts is, by itself, enough to guarantee that a validity is explicitly known:

- $\Vdash \varphi$ implies neither $\Vdash \mathrm{K}_{Ex}\,\varphi$ nor $\Vdash \mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{K}_{Ex}\,\varphi$ nor $\Vdash \mathrm{A}^{\mathrm{t}}\,\varphi \to \mathrm{K}_{Ex}\,\varphi$.

About closure under logical equivalence. Even though *awareness that* has this property, *explicit knowledge* does not need to, as the agent might be aware of a formula without being aware of a logically equivalent one. Thus,

- $\Vdash \varphi \leftrightarrow \psi$ does not imply $\Vdash \mathrm{K}_{Ex}\,\varphi \leftrightarrow \mathrm{K}_{Ex}\,\psi$.

Still, *awareness of* is the only piece that is missing. Thus, in particular, if two formulas are logically equivalent and the agent is aware of the second, then explicit knowledge of the first implies explicit knowledge of the second:

$$\Vdash \varphi \leftrightarrow \psi \quad \text{implies} \quad \Vdash \mathrm{A}^{\mathrm{o}}\,\psi \to (\mathrm{K}_{Ex}\,\varphi \to \mathrm{K}_{Ex}\,\psi).$$

About closure under modus ponens. Since *awareness that* lacks this property, explicit knowledge lacks it too:

- $\not\Vdash \mathrm{K}_{Ex}(\varphi \to \psi) \to (\mathrm{K}_{Ex}\,\varphi \to \mathrm{K}_{Ex}\,\psi)$.

[8] In a semantic setting, a modus ponens can be understood as a three-step process: conjunction introduction (from $[\![\varphi]\!]^M$ and $[\![\varphi \to \psi]\!]^M$ to $[\![\varphi \land (\varphi \to \psi)]\!]^M$), logical equivalence (from the latter to $[\![\varphi \land \psi]\!]^M$) and conjunction elimination (from the now-latter to $[\![\psi]\!]^M$).

The lack of this property is already shared by the explicit belief of [24] (defined, recall, as implicit knowledge, $\Box\varphi$, plus *awareness of*, $\mathrm{A}\,\varphi$). However, different from the mentioned proposal, here being aware of the consequent does not give the agent explicit knowledge about it:

- $\not\Vdash \mathrm{K}_{Ex}(\varphi \to \psi) \to \big((\mathrm{K}_{Ex}\,\varphi \wedge \mathrm{A}^{\mathrm{o}}\,\psi) \to \mathrm{K}_{Ex}\,\psi\big)$.

In other words, the agent might know explicitly an implication and its antecedent, and she might even entertain the consequent, and still she does not need to know explicitly that this consequent is indeed the case. In fact, assuming the agent entertains the consequent does not provide anything new. By explicitly knowing the implication, the agent is aware of it, and thus she is aware of its consequent, as *awareness of* is closed under subformulas. What the agent misses is the acknowledgement that the implication's consequence is indeed the case:[9]

$$\Vdash \mathrm{K}_{Ex}(\varphi \to \psi) \to \big((\mathrm{K}_{Ex}\,\varphi \wedge \mathrm{A}^{\mathrm{t}}\,\psi) \to \mathrm{K}_{Ex}\,\psi\big).$$

Crucially, acknowledgement can be reached via some epistemic *action* (Section 4).

3.2 *Effects of the augmentation operation*

The augmentation operation makes the neighbourhood of each world a set that is closed under supersets and contains the neighbourhood's core ($\bigcap N(w)$). It is well known (e.g. [14,Theorem 7.9]) that in the resulting model, the *augmented* model M^{*}, the operator A^{t} behaves as the standard $\Box$ does in relational models. Hence, occurrences of A^{t} under the scope of the modality $[*]$ can be understood as what the agent will acknowledge as true after she applies every possible deductive inference.[10] In this sense, the augmentation operation makes the agent's *awareness that* closed under logical consequence, and hence it can be understood as a *full deductive inference* operation over A^{t}.

***Awareness that* after full deductive inference.** As a consequence of the augmentation operation, the agent acknowledges every validity as true:

$$\Vdash \varphi \quad \text{implies} \quad \Vdash [*]\,\mathrm{A}^{\mathrm{t}}\,\varphi.$$

Moreover, her *awareness that* becomes closed under conjunction introduction,

$$\Vdash [*]\,\big((\mathrm{A}^{\mathrm{t}}\,\varphi \wedge \mathrm{A}^{\mathrm{t}}\,\psi) \to \mathrm{A}^{\mathrm{t}}(\varphi \wedge \psi)\big),$$

and also under conjunction elimination,

$$\Vdash [*]\,\big(\mathrm{A}^{\mathrm{t}}(\varphi \wedge \psi) \to \mathrm{A}^{\mathrm{t}}\,\varphi\big) \qquad \text{and} \qquad \Vdash [*]\,\big(\mathrm{A}^{\mathrm{t}}(\varphi \wedge \psi) \to \mathrm{A}^{\mathrm{t}}\,\psi\big).$$

In fact, since the augmentation operation is a total function over *ANM*s (defined for every *ANM*, and producing a single result), the last two properties can be described in a more useful way. For the first, if after the operation the agent acknowledges φ and after the operation she acknowledges ψ, then after the operation she acknowledges both and hence she acknowledges their conjunction:

$$\Vdash ([*]\,\mathrm{A}^{\mathrm{t}}\,\varphi \wedge [*]\,\mathrm{A}^{\mathrm{t}}\,\psi) \to [*]\,\mathrm{A}^{\mathrm{t}}(\varphi \wedge \psi).$$

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[9] In this, the present setting coincides with the one in [32].

[10] The study in [59] (see also [5]) already uses this known relationship between neighbourhood models and relational models. Still, the technical details are slightly different, as the referred paper works on the finite-domain case and does not take the concept of *awareness of* into account.

For the second, if after the operation the agent acknowledges a conjunction, then after the operation she acknowledges any of its conjuncts:

$$\Vdash [*]\,\mathrm{A}^{\mathrm{t}}(\varphi \land \psi) \to [*]\,\mathrm{A}^{\mathrm{t}}\,\varphi \qquad \text{and} \qquad \Vdash [*]\,\mathrm{A}^{\mathrm{t}}(\varphi \land \psi) \to [*]\,\mathrm{A}^{\mathrm{t}}\,\psi.$$

These two closure properties and the discussed closure under logical equivalence makes A^{t} closed under modus ponens *after* the augmentation operation,

$$\Vdash [*]\,\big(\mathrm{A}^{\mathrm{t}}(\varphi \to \psi) \to (\mathrm{A}^{\mathrm{t}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\psi)\big),$$

and thus, since the operation is a total function,

$$\Vdash [*]\,\mathrm{A}^{\mathrm{t}}(\varphi \to \psi) \to ([*]\,\mathrm{A}^{\mathrm{t}}\,\varphi \to [*]\,\mathrm{A}^{\mathrm{t}}\,\psi).$$

Finally, the operation also affects what the agent has acknowledged about her own *awareness of*. Recall the technical reason why the agent did not acknowledge what she entertains; she might not have acknowledged all formulas that are true in all worlds. But this ‘closure under deductive inference’ operation makes her realize that all globally true formulas are indeed the case. Hence,

$$\Vdash [*](\mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{o}}\,\varphi) \qquad \text{and} \qquad \Vdash [*](\neg\,\mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{o}}\,\varphi)$$

or, since the operation does not affect awareness sets,

$$\Vdash \mathrm{A}^{\mathrm{o}}\,\varphi \to [*]\,\mathrm{A}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{o}}\,\varphi \qquad \text{and} \qquad \Vdash \neg\,\mathrm{A}^{\mathrm{o}}\,\varphi \to [*]\,\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{o}}\,\varphi.$$

Implicit knowledge. Knowing the properties of *awareness that* after the augmentation operation, it is now time to discuss the properties of the notion of *implicit knowledge*, defined as what the agent currently entertains and will recognize as true *after* performing all possible deductive inferences, $\mathrm{K}_{Im}\,\varphi := \mathrm{A}^{\mathrm{o}}\,\varphi \land [*]\,\mathrm{A}^{\mathrm{t}}\,\varphi$.

First, the agent’s implicit knowledge does not need to contain every validity,

- $\Vdash \varphi$ does not imply $\Vdash \mathrm{K}_{Im}\,\varphi$,

the reason being that she might be unaware of some involved atoms.[11] Nevertheless, different from the *explicit knowledge* case, *awareness of* is the only missing piece; the agent knows *implicitly* any validity she is currently entertaining:

$$\Vdash \varphi \quad \text{implies} \quad \Vdash \mathrm{A}^{\mathrm{o}}\,\varphi \to \mathrm{K}_{Im}\,\varphi.$$

Similarly, the agent’s implicit knowledge is not closed under logical equivalence,

- $\Vdash \varphi \leftrightarrow \psi$ does not imply $\Vdash \mathrm{K}_{Im}\,\varphi \leftrightarrow \mathrm{K}_{Im}\,\psi$,

the only reason being that the *awareness of* requirement might fail. Thus,

$$\Vdash \varphi \leftrightarrow \psi \quad \text{implies} \quad \Vdash (\mathrm{K}_{Im}\,\varphi \land \mathrm{A}^{\mathrm{o}}\,\psi) \to \mathrm{K}_{Im}\,\psi.$$

Finally, implicit knowledge is closed under both conjunction introduction and conjunction elimination. For the first, (i) after the augmentation, *awareness that* has such property (use here the alternative version of this), and (ii) *awareness of* is closed under superformulas :

$$\Vdash (\mathrm{K}_{Im}\,\varphi \land \mathrm{K}_{Im}\,\psi) \to \mathrm{K}_{Im}(\varphi \land \psi).$$

[11] This differs from the same notion in [24], where the agent knows implicitly every validity.

Analogously, for the second: (i) after the operation, *awareness that* has the property (use its alternative version), and (ii) *awareness of* is closed under subformulas :

$$\Vdash \mathrm{K}_{Im}(\varphi \wedge \psi) \rightarrow \mathrm{K}_{Im}\,\varphi \qquad \text{and} \qquad \Vdash \mathrm{K}_{Im}(\varphi \wedge \psi) \rightarrow \mathrm{K}_{Im}\,\psi.$$

The last three validity statements, together with *awareness of*'s closure under subformulas, tell us that implicit knowledge is closed under modus ponens:

$$\Vdash \mathrm{K}_{Im}(\varphi \rightarrow \psi) \rightarrow (\mathrm{K}_{Im}\,\varphi \rightarrow \mathrm{K}_{Im}\,\psi).$$

Observe how, while the implicit knowledge of [24] contains all validities and is closed under modus ponens (i.e. is closed under logical consequence), implicit knowledge as defined here might not contain all validities and, despite being closed under modus ponens, it does not need to be closed under logical equivalence. More precisely, while implicit knowledge in [24] is the agent's 'semantic' information (given by the modal operator $\Box$), here it is the closure under modus ponens of what the agent has acknowledged as true ($[*]\,\mathrm{A}^{\mathrm{t}}\,\varphi$) and is currently entertaining ($\mathrm{A}^{\mathrm{o}}\,\varphi$); it is the closure under modus ponens of what the agent knows explicitly.

This point highlights the crucial difference between the understanding of implicit knowledge in this proposal and in [24]. In the latter, what is needed for implicit knowledge to be explicit is for the agent to be aware of the given formula. However, in this text, knowing a formula implicitly already implies the agent is aware of it. Here, what is needed for implicit knowledge to become explicit is then not an act of awareness raising; what is needed is an act of deductive inference.

Moorean phenomena. In proposals dealing with implicit and explicit knowledge, a particular property is recurrent: explicit knowledge is also implicit. In [24], this follows from the fact that explicit knowledge is defined as the implicit knowledge that satisfies an additional requirement (*awareness of*); in settings distinguishing explicit information from implicit one by means of deductive reasoning (e.g. [41]), this follows from the fact that deductive inference is monotone.

This property, seemingly not only natural but rather essential, fails here:

- $\not\Vdash \mathrm{K}_{Ex}\,\varphi \rightarrow \mathrm{K}_{Im}\,\varphi$.

The reason is [59] that what the agent has acknowledged as true at some stage does not need to be acknowledged as true after the augmentation operation, i.e.

FACT 1
$\not\Vdash \mathrm{A}^{\mathrm{t}}\,\varphi \rightarrow [*]\,\mathrm{A}^{\mathrm{t}}\,\varphi$.

PROOF. Take the set of atomic propositions $\{p, q\}$ and the formula $\varphi := \neg\,\mathrm{A}^{\mathrm{t}}\,q$, and consider a four-worlds model $M = \langle W = \{w_1, w_2, w_3, w_4\}, N, V, \varnothing\rangle$ with an empty atomic awareness set, valuation given by $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_1, w_3\}$, and neighbourhood function such that

$$N(w_1) := \{\{w_1, w_2\}, \{w_1, w_3, w_4\}, W\}, \qquad N(w_2) = N(w_3) = N(w_4) := \varnothing.$$

From $[\![\mathrm{A}^{\mathrm{t}}\,q]\!]^M = \varnothing$ (no neighbourhood contains $[\![q]\!]^M = \{w_1, w_3\}$) it follows that $[\![\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^M = W$; hence, $[\![\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^M \in N(w_1)$ and therefore $w_1 \in [\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^M$. However, observe the neighbourhood function of the augmented model M^*:

$$N^*(w_1) = \{U \subseteq W \mid w_1 \in U\}, \qquad N^*(w_2) = N^*(w_3) = N^*(w_4) = \{W\}.$$

Note how $[\![\mathrm{A}^{\mathrm{t}}\,q]\!]^{M^*} = \{w_1\}$ (only w_1's neighbourhood includes $[\![q]\!]^{M^*} = \{w_1, w_3\}$), so $[\![\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^{M^*} = \{w_2, w_3, w_4\}$. Then, $[\![\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^{M^*} \notin N^*(w_1)$ so $w_1 \notin [\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^{M^*}$, i.e. $w_1 \notin [\![[*]\,\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}}\,q]\!]^M$. $\Box$

Thus, while in M the agent has acknowledged $\neg\,\mathrm{A^t}\, q$ as true at w_1 (i.e. $w_1 \in [\![\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M$), the operation changes this: in M^*, the agent has not acknowledged $\neg\,\mathrm{A^t}\, q$ as true at w_1 (i.e. $w_1 \notin [\![[*]\,\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M$). One just needs to make the agent aware of the involved formula $\neg\,\mathrm{A^t}\, q$ (e.g. take $A := \{p, q\}$) to obtain a model (M) and a world (w_1) in which the agent knows a formula ($\neg\,\mathrm{A^t}\, q$) explicitly,

$$
\begin{aligned}
w_1 \in [\![\mathrm{K}_{Ex}\,\neg\,\mathrm{A^t}\, q]\!]^M &= [\![\mathrm{A^o}\,\neg\,\mathrm{A^t}\, q \;\wedge\; \mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M \\
&= [\![\mathrm{A^o}\,\neg\,\mathrm{A^t}\, q]\!]^M \cap [\![\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M ,
\end{aligned}
$$

and yet she does not know it implicitly,

$$
\begin{aligned}
w_1 \notin [\![\mathrm{K}_{Im}\,\neg\,\mathrm{A^t}\, q]\!]^M &= [\![\mathrm{A^o}\,\neg\,\mathrm{A^t}\, q \;\wedge\; [*]\,\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M \\
&= [\![\mathrm{A^o}\,\neg\,\mathrm{A^t}\, q]\!]^M \cap [\![[*]\,\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q]\!]^M .
\end{aligned}
$$

So, is there some fundamental problem with the current proposal? To answer this, first note how the 'explicit is implicit' property does not always fail. In fact, it holds for a large class of formulas, including not only the purely propositional ones, but also all those whose truth set *does not shrink* as a consequence of the augmentation operation.

PROPOSITION 1

$$\Vdash \varphi \rightarrow [*]\,\varphi \quad \text{implies} \quad \Vdash \mathrm{K}_{Ex}\,\varphi \rightarrow \mathrm{K}_{Im}\,\varphi.$$

PROOF. That of [59, Proposition 2] plus *awareness of*. □

The counterexample provides a hint on why the rest of the formulas fail; they express not ontic facts, but rather *epistemic* ones and, in particular *negative awareness that* situations. Indeed, $\neg\,\mathrm{A^t}\, q$ expresses that the agent has not acknowledged q as true, and then $\mathrm{A^t}\,\neg\,\mathrm{A^t}\, q$ says that the agent has acknowledged this. In other words, and considering her *awareness of*, the agent knows explicitly that she does not know q explicitly.

However, she might have enough information to 'extract' what she currently knows she does not have. In the provided model (with $A := \{p, q\}$), at w_1, she knows explicitly both p and $p \rightarrow q$ (she is aware of all atoms, hence of all formulas, and both $[\![p]\!]^M = \{w_1, w_3\}$ and $[\![p \rightarrow q]\!]^M = \{w_1, w_3, w_4\}$ are in $N(w_1)$); thus, after deductive reasoning, she will realize that q is indeed the case, hence knowing q explicitly. But then, she will automatically stop acknowledging (and thus stop knowing explicitly) that she did not know q explicitly. In other words, she might know explicitly that she does not know q, but such *high-order* knowledge will be gone once she gets to know that q is indeed the case.

The reason for the failure of the 'explicit is implicit' property is that the agent has knowledge not only about propositional facts but also about her own (and eventually other agents') knowledge. This knowledge (semantically, the neighbourhood function) changes through the augmentation operation; the agent might know something explicitly at some point, and yet not know it explicitly afterwards (semantically, the *awareness that* component is the key; we might have $U \in N(w)$ with $U = [\![\varphi]\!]^M$ for some φ, but even though this implies $U \in N^*(w)$, nothing guarantees $U = [\![\varphi]\!]^{M^*}$). This is nothing but an instance of the so called 'Moorean phenomena' in *DEL*, which occurs when an epistemic action ‘invalidates’ itself. In its best known incarnation, this phenomenon appears as formulas that become false after being truthfully announced [19, 37]; here, it appears as formulas that stop being known after deductive inference.

3.3 Alternatives for representing the basic notions

The basic idea behind this proposal is to define the concept of explicit knowledge as *awareness of* plus *awareness that*. These two concepts have been studied in the literature, sometimes under different names. The remaining part of this section recalls some of those proposals, discussing their relationship with the current one.

***Awareness of*.** As mentioned, this concept describes what the agent entertains, and it does not imply any attitude in favour or against the given formula. Here, the concept is semantically represented by means of a set of atomic propositions (in the style of [24]), indicating the atoms the agent is currently aware of.

There are other alternatives for representing such notion. One of them is the treatment of the concept of *topics* in [12] (cf. [11]), with the topic of a sentence being what the sentence is *about* ([62]; cf. [27]). Such proposal assumes an underlying set of topics $\mathcal{T}$, a topic *fusion* operation $\oplus : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ (intended to make topics part of bigger topics), and a topic *parthood* relation $\leq\ \subseteq (\mathcal{T} \times \mathcal{T})$ (defining a ‘subtopic’ partial order). The fusion operator $\oplus$ is assumed to be $(1_\oplus)$ idempotent ($t \oplus t = t$ for all $t \in \mathcal{T}$), $(2_\oplus)$ commutative ($t \oplus s = s \oplus t$ for all $t, s \in \mathcal{T}$) and $(3_\oplus)$ associative ($(t \oplus s) \oplus r = t \oplus (s \oplus r)$ for all $t, s, r \in \mathcal{T}$) ; the parthood relation $\leq$ is assumed to be $(1_\leq)$ reflexive ($t \leq t$), $(2_\leq)$ antisymmetric ($t \leq s$ and $s \leq t$ imply $t = s$) and $(3_\leq)$ transitive ($t \leq s$ and $s \leq r$ imply $t \leq r$) , with the set of *atomic* topics $\mathcal{T}_0$ containing exactly those topics lying at the bottom of the $\leq$-ordering. The link from topics to formulas is given by a function $T : \mathcal{L} \rightarrow \mathcal{T}$, which assigns an atomic topic to atomic propositions (i.e. $T(p) \in \mathcal{T}_0$) and then uses the fusion operator to assign topics to more complex formulas ($T(\varphi) := \bigoplus_{p \in \mathrm{at}(\varphi)} T(p)$).

The theory of topics is a generalization of *awareness of* based on atoms, in which take a topic to be a set of atoms ($\mathcal{T} := \wp(\mathtt{P})$), topic fusion as the set-theoretical union ($\oplus := \cup$) and the parthood ordering as the subset ordering ($\leq\ :=\ \subseteq$). This satisfies the indicated properties: for any $Q, R, S \in \wp(\mathtt{P})$, $(1_\oplus)$ $Q \cup Q = Q$ (idempotency), $(2_\oplus)$ $Q \cup R = R \cup Q$ (commutativity) and $(3_\oplus)$ $(Q \cup R) \cup S = Q \cup (R \cup S)$ (associativity) ; moreover, we also have $(1_\leq)$ $Q \subseteq Q$ (reflexivity), $(2_\leq)$ $Q \subseteq R$ and $R \subseteq Q$ imply $Q = R$ (antisymmetry) and $(3_\leq)$ $Q \subseteq R$ and $R \subseteq S$ imply $Q \subseteq S$ (transitivity) . Thus, taking $T(\varphi) := \mathrm{at}(\varphi)$ fulfils the desiderata.

There are also more semantic options for representing *awareness of*. One alternative within the possible worlds realm is to use an equivalence relation, the *issue relation*, which creates a partition of the domain, standing for the ‘degree of abstraction’ the agent has according to the issues that are currently under consideration (i.e. the possibilities she is currently entertaining). For example, consider a model with four possible worlds, exactly the four possible valuations for atoms in $\{p, q\}$. A ‘blissfully ignorant’ agent (one not knowing anything about these atoms, and yet being aware of both and also fully introspective about her own knowledge) sees the model as four different possibilities, i.e. four partitions, each one of them containing a single possible world. However, an agent only aware of p is not considering the further differences q can make. Thus, she sees the model as only two partitions: one in which p holds (hence containing the worlds $[p, q]$ and $[p, \neg q]$), and another in which p fails (hence containing the worlds $[\neg p, q]$ and $[\neg p, \neg q]$); something analogous happens when an agent is only aware of q. Finally, an agent unaware of both p and q sees a single partition containing all possible worlds.

The issues under consideration do not need to be atoms; an agent pondering whether $p \rightarrow q$ holds will see two partitions, one in which the implication is true (with worlds $[p, q]$, $[\neg p, q]$ and $[\neg p, \neg q]$) and another in which the implication is false (with world $[p, \neg q]$). This relational representation of the issues at stake, used in logical frameworks in, e.g., [6, 8, 31], is already present in different places dealing with the notion of questions, ranging from linguistics [30] to learning theory [39].

The present proposal has chosen a syntactic representation (a set of atoms) because it allows a finer distinction. For example, there are models in which two different atoms happen to be true in exactly the same possible worlds; in such cases, an issue relation would not allow agents that are aware of the first but not of the second, something a set of atoms can.

Awareness that. Here, *awareness that* is intended to describe what the agent has accepted (i.e. acknowledged) as true. In frameworks dealing with non-omniscient agents without using the notion of *awareness of* (e.g. [3, 22, 38, 40, 42, 59]), this is typically understood as explicit knowledge: what the agent actually has.

This notion is usually represented syntactically, by a set of formulas (among the ones listed before: [22, 38, 40]). When no further constrain is imposed in such sets (as, e.g. the closure under logical consequence required by belief sets in the Alchourrón, Gärdenfors, and Makinson (AGM) belief revision of [1]), this representation avoids any form of logical omniscience as, e.g. no validity is required to be an element of the set, and having an implication and its antecedent does not guarantee the presence of the consequent.[12, 13] Still, such settings can be criticized for being *too* fine-grained as, e.g., the agent might know/believe $\varphi \vee \psi$ without knowing/believing $\psi \vee \varphi$.

Other options have a more semantic flavour. For example, [42] relies on *situations*, semantic entities that might support the truth or falsity of a formula, but can also support neither and even both (cf. the truthmaker semantics of [28]). A more 'traditional' approach is the logic of local reasoning of [24], which uses standard possible worlds but follows the idea that, rather than having a single system of beliefs, real agents have a number of distinct systems, with different ones working on different contexts (the *fragmentation/compartmentalization* of [15, 23]).

The present proposal has relied on the use of neighbourhood semantics, thus using a purely semantic representation that still achieves the goal of making the agent non-ideal (see Section 3.1).[14] Being a purely semantic representation, this proposal's *awareness that* cannot distinguish between logically equivalent formulas. Still, this 'problem' is taken care of by the additional syntactic component the notion of explicit knowledge requires: *awareness of*.

Before closing, it is worthwhile mentioning the existence of other options, which represent a notion intuitively similar to *awareness that*, but do so by relying on further epistemic concepts. The syntactic notion of justification in *justification logics* has been briefly mentioned, but there are also (purely semantic) alternatives in which the agent's knowledge/belief relies on concepts as *evidence* [9, 45] and *arguments* [52–54]. In such settings, the precise interplay between the different pieces of evidence (resp., the different arguments) and the way they define knowledge and/or beliefs determine the closure properties of the latter notions. Thus, under adequate relaxations, they might provide suitable frameworks for dealing with the explicit knowledge of non-ideal agents (see, e.g. [4, 13]).

[12] An *impossible world* [36, 48, 49] can be understood as an arbitrary set of formulas.

[13] The logics of justification [2, 3] can be also seen as using a syntactic device for distinguishing what the agent actually has from what she can eventually obtain, the difference being that this syntactic component, understood as the formula's *justification*, has a whole justification calculus associated to it.

[14] In fact, the referred *logic of local reasoning* [24] is equivalent to a neighbourhood semantics in which each neighbourhood is assumed to be non-empty and also closed under supersets, thus making the knowledge of the agent closed under conjunction elimination.

4 Epistemic actions

The provided framework allows the representation of the epistemic state of non-ideal agents. Still, some authors ([10, 20, 22], among others) have argued that solutions of this form are not completely adequate. First, an agent's (explicit) knowledge can be weakened in many ways, and there is no clear method to decide which restrictions are the appropriate ones. Second, such approaches do not look at the heart of the matter; they still describe the agent's epistemic state at a single stage, without looking at how such state is reached. What is needed is, then, not only a representation of non-ideal agents, but also a representation of the *actions* through which such an agent can change her epistemic state.

Recognizing the role of epistemic actions is crucial. First, asking for additional requirements for a notion to be called explicit knowledge might make the agent non-omniscient, but providing the epistemic actions that allow her to fulfil such requirements guarantees that she will not be defective or ignorant (in other words, she will be *rational*). Second, introducing the actions that lead to 'omniscient' states may demystify them. In Conan Doyle's detective stories, the explanation offered at the end turns Holmes' 'magical powers' into a sequence of observations and deductive acts, making the whole procedure *'elementary, my dear Watson'*. This section defines some of these actions, discussing also their basic properties.

Becoming aware of. Actions of changes in *awareness of* can be semantically represented by model operations that change the agent's atomic awareness set, leaving other model components unaffected (see, e.g. [10, 16]). The first one, becoming aware of a given formula χ, boils down to adding χ's atoms to this set.

DEFINITION 3 (*Becoming aware of* operation).
Let $M = \langle W, N, V, A \rangle$ be an *ANM*; let χ be a formula in $\mathcal{L}$. The model $M^{+\chi} = \left\langle W, N, V, A^{+\chi} \right\rangle$ differs from M only in its atomic awareness set, which is extended with the atoms of χ:

$$A^{+\chi} := A \cup \mathrm{at}(\chi).$$

For reasoning about the effects of such action, the language $\mathcal{L}_{[+\chi]}$ extends $\mathcal{L}$ with a *becoming aware of* modality $[+\chi]$ for each formula χ; if φ is a formula in this extended language, then so is $[+\chi]\,\varphi$, a formula read as '*φ is the case after the agent becomes aware of χ*'.[15] The modalities' semantic interpretation relies on the model the *becoming aware of* operation yields:[16]

$$[\![[+\chi]\,\varphi]\!]^{M} := [\![\varphi]\!]^{M^{+\chi}}.$$ [15]

Basic properties. The first property of this operation is simply a 'sanity check'; after becoming aware of a formula χ, the agent is aware of it:

$$\Vdash [+\chi]\,\mathrm{A^o}\,\chi.$$

Moreover, *awareness of* is based on atomic propositions, and thus can be seen as the language the agent has at her disposal. Then, becoming aware of a formula χ also makes the agent not only aware of all χ's atoms, but also aware of all formulas built from their combination. In particular, after becoming aware of a formula, the agent will be aware of all the formula's subformulas. For

[15] Define $\mathrm{at}([+\chi]\,\varphi) := \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$.

[16] Defining a 'diamond' *becoming aware of* modality in the standard way, $\langle +\chi \rangle\,\varphi := \neg\,[+\chi]\,\neg\varphi$, implies $[\![\langle +\chi \rangle\,\varphi]\!]^{M} = [\![\varphi]\!]^{M^{+\chi}}$; under this definition, $[+\chi]\,\varphi$ and $\langle +\chi \rangle\,\varphi$ are logically equivalent.

example,

$$\Vdash [+\neg\varphi]\, \mathrm{A^o}\, \varphi, \qquad \Vdash [+\mathrm{A^o}\, \varphi]\, \mathrm{A^o}\, \varphi,$$
$$\Vdash [+(\varphi \wedge \psi)](\mathrm{A^o}\, \varphi \wedge \mathrm{A^o}\, \psi), \qquad \Vdash [+\mathrm{A^t}\, \varphi]\, \mathrm{A^o}\, \varphi,$$
$$\Vdash [+[*]\, \varphi]\, \mathrm{A^o}\, \varphi.$$

More generally, after becoming aware of χ, the agent will be aware of φ if and only if she was already aware of all atoms that are in φ but not in χ:[17]

$$\Vdash [+\chi]\, \mathrm{A^o}\, \varphi \leftrightarrow \bigwedge_{p \in \mathrm{at}(\varphi) \setminus \mathrm{at}(\chi)} \mathrm{A^o}\, p.$$

Its effect on explicit knowledge. In other proposals for actions with similar spirit (see the mentioned references), becoming aware of an implicitly known formula gives the agent explicit knowledge about it. This is not the case here; the action makes the agent aware of the formula, but it does not guarantee that she will acknowledge it as true afterwards (the neighbourhood function is not affected by the operation). Even assuming that the agent has implicit knowledge of the formula before the *becoming aware of* action is not enough, as this does not say anything about what the agent has acknowledged as true at such stage. Indeed, implicit knowledge only makes the agent aware of the formula (so the *becoming aware of* action has no effect), but still the agent would need to do a further ‘acknowledgement’ step in order to make the formula part of her explicit knowledge.

- $\not\Vdash \mathrm{K}_{Im}\, \chi \rightarrow [+\chi]\, \mathrm{A^t}\, \chi$,
- $\not\Vdash \mathrm{K}_{Im}\, \chi \rightarrow [+\chi]\, \mathrm{K}_{Ex}\, \chi$.

Thus, becoming aware of a formula is not the same as recognizing that it holds.

Given the stated $\Vdash [+\chi]\, \mathrm{A^o}\, \chi$ and the fact that explicit knowledge is defined as awareness of plus awareness that, one might think that, if the agent has already acknowledged a given χ as true, then after becoming aware of it she would know it explicitly. This indeed is the case for propositional formulas

$$\Vdash \mathrm{A^t}\, \gamma \rightarrow [+\gamma]\, \mathrm{K}_{Ex}\, \gamma \qquad \text{for } \gamma \text{ a propositional formula}$$

and even for some non-propositional ones.[18] However, the formula is not valid in general, as the operation might not preserve acknowledgement of truth.

FACT 2
$\not\Vdash \mathrm{A^t}\, \chi \rightarrow [+\chi]\, \mathrm{A^t}\, \chi$.

PROOF. Take the set of atomic propositions $\{p\}$ and the formula $\varphi := \neg\, \mathrm{A^o}\, p$; consider the one-world model $M = \langle W = \{w\}, N, V, \varnothing \rangle$ with an empty atomic awareness set and a neighbourhood function given by $N(w) = \{W\}$ (the atomic valuation is not relevant). Note how $[\![\mathrm{A^o}\, p]\!]^M = \varnothing$ (since $p \notin A$), and therefore $[\![\neg\, \mathrm{A^o}\, p]\!]^M = W = \{w\}$; then,

$$[\![\mathrm{A^t} \neg\, \mathrm{A^o}\, p]\!]^M = \{u \in W \mid [\![\neg\, \mathrm{A^o}\, p]\!]^M \in N(u)\} = \{u \in W \mid W \in N(u)\} = \{w\}$$

so $w \in [\![\mathrm{A^t} \neg\, \mathrm{A^o}\, p]\!]^M$; at world w in model M, the agent has acknowledged that she is not aware of p. Now, the *becoming aware of* operation with $\neg\, \mathrm{A^o}\, p$ yields an *ANM* $M^{+\neg\, \mathrm{A^o}\, p}$ whose atomic

[17] If there are no such atoms (if $\mathrm{at}(\varphi) \subseteq \mathrm{at}(\chi)$), the formula’s right-hand side collapses to $\top$; after becoming aware of χ, the agent will be aware of every φ built only from atoms in χ.

[18] More precisely, the validity holds exactly for those formulas φ whose truth set is not affected by the operation, i.e. for every $\varphi \in \mathcal{L}$ for which $[\![\varphi]\!]^M = [\![\varphi]\!]^{M^{+\chi}}$.

awareness set is $A^{+\neg \mathrm{A}^{\mathrm{o}} p} = \{p\}$; thus, $[\![\mathrm{A}^{\mathrm{o}}\, p]\!]^{M^{+\neg \mathrm{A}^{\mathrm{o}} p}} = W$, and therefore $[\![\neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M^{+\neg \mathrm{A}^{\mathrm{o}} p}} = \varnothing$. Hence,

$$[\![\mathrm{A}^{\mathrm{t}} \neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M^{+\neg \mathrm{A}^{\mathrm{o}} p}} = \{u \in W \mid [\![\neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M^{+\neg \mathrm{A}^{\mathrm{o}} p}} \in N(u)\} = \{u \in W \mid \varnothing \in N(u)\} = \varnothing$$

so $[\![[+\neg\, \mathrm{A}^{\mathrm{o}}\, p]\, \mathrm{A}^{\mathrm{t}} \neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M} = \varnothing$, i.e. $w \notin [\![[+\neg\, \mathrm{A}^{\mathrm{o}}\, p]\, \mathrm{A}^{\mathrm{t}} \neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M}$; at world w in the model that results from the agent becoming aware of $\neg\, \mathrm{A}^{\mathrm{o}}\, p$, the agent has *not* acknowledged that she is not aware of p. Summarizing,

$$w \notin [\![\mathrm{A}^{\mathrm{t}} \neg\, \mathrm{A}^{\mathrm{o}}\, p \rightarrow [+\neg\, \mathrm{A}^{\mathrm{o}}\, p]\, \mathrm{A}^{\mathrm{t}} \neg\, \mathrm{A}^{\mathrm{o}}\, p]\!]^{M}.$$

□

The counterexample used to prove this fact shows again a Moorean effect; the agent might have accepted a formula as true, but a model operation might change the set of worlds in which the formula is true (in the counterexample, from W to $\varnothing$). Then, even though the neighbourhood function remains the same, the formulas the agent has acknowledged as true might change.

Becoming unaware of. An agent can also become *unaware of* a given formula; such action can be represented by shrinking her awareness set.

DEFINITION 4 (*Becoming unaware of* operation).
Let $M = \langle W, N, V, A\rangle$ be an *ANM*; let χ be a formula in $\mathcal{L}$. The model $M^{-\chi} = \langle W, N, V, A^{-\chi}\rangle$ differs from M only in its atomic awareness set, from which the atoms in χ have been removed:

$$A^{-\chi} := A \setminus \mathrm{at}(\chi).$$

On the syntactic side, $\mathcal{L}_{[-\chi]}$ extends $\mathcal{L}$ with a *becoming unaware of* modality $[-\chi]$ for every formula χ. Formulas of the form $[-\chi]\,\varphi$, read as *'after the agent becomes unaware of* χ, φ *is the case'*,[19] are interpreted in the new model:[20]

$$[\![[-\chi]\,\varphi]\!]^{M} := [\![\varphi]\!]^{M^{-\chi}}.$$ [19]

Basic properties. The operation passes the 'sanity check':

$$\Vdash [-\chi]\,\neg\, \mathrm{A}^{\mathrm{o}}\, \chi \qquad \text{for any}\, \chi \text{ such that}\, \mathrm{at}(\chi) \neq \varnothing.$$

The restriction is because the agent is always aware of every formula that does not involve atoms (e.g. $\top, \neg\top, \top \wedge \top$); thus, she cannot become unaware of any of them. Still, the agent becomes unaware of the given χ by becoming unaware of *all* its atoms (see below for a weaker alternative); thus, she also becomes unaware of *every formula involving atoms in* χ:[21]

$$\Vdash [-\chi]\,\neg\, \mathrm{A}^{\mathrm{o}}\, \varphi \qquad \text{for any } \varphi \text{ such that } \mathrm{at}(\chi) \cap \mathrm{at}(\varphi) \neq \varnothing.$$ [20]

Its effect on explicit knowledge. Different from the *becoming aware of* case, the *becoming unaware of* operation behaves as expected in the sense that, after becoming unaware of a given formula (involving at least one atom), the agent will not have explicit knowledge about it, the reason being that, in such case, the agent will become unaware of the given formula, as observed before.

$$\Vdash [-\chi]\,\neg\, \mathrm{K}_{Ex}\, \chi \qquad \text{for any } \varphi \text{ such that } \mathrm{at}(\chi) \neq \varnothing,$$

[19] Define $\mathrm{at}([-\chi]\,\varphi) := \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$.
[20] Just as before, defining $\langle -\chi\rangle\, \varphi := \neg\, [-\chi]\, \neg\varphi$ implies $\Vdash [-\chi]\,\varphi \leftrightarrow \langle -\chi\rangle\, \varphi$.
[21] In particular, by becoming unaware of χ, the agent becomes unaware of all χ's subformulas.

A weaker form of becoming unaware of. The just defined action has very strong effects; after becoming unaware of χ, the agent also becomes unaware of every formula involving atoms in χ. Still, it is possible to provide an alternative operation, and then use it to define a weaker form of becoming unaware of.

DEFINITION 5 (*Becoming unaware of* operation).
Let $M = \langle W, N, V, A \rangle$ be an *ANM*; take $\mathtt{Q} \subseteq \mathtt{P}$. The model $M^{-\mathtt{Q}} = \langle W, N, V, A^{-\mathtt{Q}} \rangle$ differs from M only in its atomic awareness set, from which the atoms in $\mathtt{Q}$ have been removed:

$$A^{-\mathtt{Q}} := A \setminus \mathtt{Q}.$$

Syntactically, $\mathcal{L}_{[-\chi]}$ extends $\mathcal{L}$ with *becoming unaware of* modalities $[-\mathtt{Q}]$ for $\mathtt{Q} \subseteq \mathtt{P}$, with $[-\mathtt{Q}]\,\varphi$ read as *'after the agent becomes unaware of atoms in* $\mathtt{Q}, \varphi$ *is the case'*[22] and semantically interpreted as[23]

$$[\![[-\mathtt{Q}]\,\varphi]\!]^M := [\![\varphi]\!]^{M^{-\mathtt{Q}}}.$$ [22]

Obviously, $\Vdash [-\chi]\,\varphi \leftrightarrow [-\,\mathrm{at}(\chi)]\,\varphi$.

The operator $[-\mathtt{Q}]$ can be used to define another modality for an action of becoming unaware of a formula by quantifying over its non-empty set of atoms:

$$[\sim\chi]\,\varphi := \bigwedge_{\{\mathtt{Q}\subseteq \mathrm{at}(\chi)\ \mid\ \mathtt{Q}\neq\varnothing\}} [-\mathtt{Q}]\,\varphi.$$

In words, φ is the case after the agent becomes unaware of χ if and only if it holds after the agent becomes unaware of any non-empty set of atoms of χ. This new modality also passes the 'sanity check', as

$$\Vdash [\sim\chi]\,\neg\,\mathrm{A^o}\,\chi \qquad \text{for any } \chi \text{ such that } \mathrm{at}(\chi) \neq \varnothing.$$

Now, if χ involves atoms, the modality $[\sim\chi]$ is even stronger than $[-\chi]$. First, for such χs, the formula $[-\,\mathrm{at}(\chi)]\,\varphi$ occurs in the conjunction defining $[\sim\chi]$; thus,

$$\Vdash [\sim\chi]\,\varphi \rightarrow [-\chi]\,\varphi.$$

Second,

- $\not\Vdash [-\chi]\,\varphi \rightarrow [\sim\chi]\,\varphi$,

as φ holding after the agent becomes unaware of all atoms in χ does not guarantee that φ holds after she becomes unaware of any of its non-empty subsets.[24]

However, for similar χs (i.e. for χs involving atoms), the dual $\langle\sim\chi\rangle\,\varphi := \neg\,[\sim\chi]\,\neg\varphi$ is weaker. From its definition it follows that

$$\Vdash \langle\sim\chi\rangle\,\varphi \leftrightarrow \bigvee_{\{\mathtt{Q}\subseteq \mathrm{at}(\chi)\ \mid\ \mathtt{Q}\neq\varnothing\}} [-\mathtt{Q}]\,\varphi.$$

Then, first, since $[-\,\mathrm{at}(\chi)]\,\varphi$ occurs in the disjunction defining $\langle\sim\chi\rangle$,

$$\Vdash [-\chi]\,\varphi \rightarrow \langle\sim\chi\rangle\,\varphi.$$

[22] Define $\mathrm{at}([-\mathtt{Q}]\,\varphi) := \mathtt{Q} \cup \mathrm{at}(\varphi)$.
[23] Again, $\langle-\mathtt{Q}\rangle\,\varphi := \neg\,[-\mathtt{Q}]\,\neg\varphi$ implies $\Vdash [+\mathtt{Q}]\,\varphi \leftrightarrow \langle+\mathtt{Q}\rangle\,\varphi$.
[24] Take $\varphi := \bigwedge_{p\in \mathrm{at}(\chi)} \neg\,\mathrm{A^o}\,p$, stating that the agent is unaware of *all* atoms in χ. Clearly, it holds when all atoms are removed, but fails when some of them remain.

Second, the fact that φ holds after removing some non-empty subset of atoms of χ from the agent's awareness set does not guarantee that it also holds after removing the specific subset $\mathrm{at}(\chi)$.[25] Thus,

- $\not\Vdash \langle\sim\chi\rangle\,\varphi \rightarrow [-\chi]\,\varphi$.

For a concrete case illustrating the relationship between these two modalities, take $\chi := p \wedge q$. The formula $[-(p \wedge q)](\mathrm{A}^{\mathrm{o}}\, p \vee \mathrm{A}^{\mathrm{o}}\, q)$ is not satisfiable, as the operation removes both atoms p and q. However, $\langle\sim(p \wedge q)\rangle(\mathrm{A}^{\mathrm{o}}\, p \vee \mathrm{A}^{\mathrm{o}}\, q)$ is, as there is a way to remove at least one atom from $p \wedge q$ that yields a model in which the agent is still aware of at least one of such atoms.

Despite describing a weaker form of becoming unaware of a formula, the modality $\langle\sim\chi\rangle$ works properly. Each one of the disjuncts that define it, $[-\mathsf{Q}]$ for Q a non-empty subset of $\mathrm{at}(\chi)$, guarantees that afterwards the agent will not be aware of χ ($\Vdash [-\mathsf{Q}]\,\neg\,\mathrm{A}^{\mathrm{o}}\,\chi$ for every such Q, as the agent becomes unaware of at least one of χ's atoms); thus, after any of them, the agent will not know χ explicitly,

$$\Vdash \langle\sim\chi\rangle\,\neg\,\mathrm{K}_{Ex}\,\chi \qquad \text{for any } \varphi \text{ such that } \mathrm{at}(\chi) \neq \varnothing.$$

Modus ponens. The two previous actions describe changes in what the agent entertains. Analogously, one can also represent actions that change what the agent has acknowledged as true. The action of full deductive closure, represented by the augmentation operation, is already one of them; yet, it can be argued that such 'idealized' action is not best suited for the non-ideal agent this proposal aims to describe. A more appropriate action would be one that, instead of extracting at once all the consequences of what the agent currently has, extracts only a single piece of them. One could define different actions with such behaviour, as one performing a single act of conjunction introduction, or one performing a single conjunction elimination step. This subsection looks at the representation of a more basic inference rule: *modus ponens*.[26]

DEFINITION 6 (*Modus ponens* operation).
Let $M = \langle W, N, V, A\rangle$ be an *ANM*; let $\xi \rightarrow \chi$ be an implication in $\mathcal{L}$. The model $M^{\xi\hookrightarrow\chi} = \langle W, N^{\xi\hookrightarrow\chi}, V, A\rangle$ differs from M only in the neighbourhood function, extending that of M with the truth set of χ at M for those worlds where the agent knows explicitly both ξ and $\xi \rightarrow \chi$:

$$N^{\xi\hookrightarrow\chi}(w) := \begin{cases} N(w) \cup \{[\![\chi]\!]^M\} & \text{if } w \in [\![\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi]\!]^M \\ N(w) & \text{otherwise.} \end{cases}$$

The operation adds $[\![\chi]\!]^M$ only to the neighbourhood of those worlds in which the agent knows explicitly both the implication $\xi \rightarrow \chi$ and its antecedent ξ. In this way, the operation truly represents an act of modus ponens inference.

For the syntax, $\mathcal{L}_{[\hookrightarrow]}$ extends $\mathcal{L}$ with a *modus ponens* modality $[^{\xi}\!\hookrightarrow_{\chi}]$ for each implication $\xi \rightarrow \chi$, with $[^{\xi}\!\hookrightarrow_{\chi}]\,\varphi$ read as '*φ is the case after the agent performs a modus ponens step with $\xi \rightarrow \chi$*'.[27] Its interpretation is given by

$$[\![[^{\xi}\!\hookrightarrow_{\chi}]\,\varphi]\!]^M = [\![\varphi]\!]^{M^{\xi\hookrightarrow\chi}}.$$

[25] Take $\varphi := \bigvee_{p\in\mathrm{at}(\chi)} \mathrm{A}^{\mathrm{o}}\, p$, stating that the agent is aware of at least one of χ's atoms; it holds when χ has at least two atoms and only one of them is removed, but it fails when all atoms are discarded.

[26] Other proposals with operations with the same spirit: the deductive inferences of [5, 55, 56, 58, 59], and the belief-based inference of [60].

[27] Define $\mathrm{at}([^{\xi}\!\hookrightarrow_{\chi}]\,\varphi) := \mathrm{at}(\xi) \cup \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$.

Note the way the precondition of action is handled. Typically, the operation always 'adds' the new information, and it is the semantic interpretation of its associated modality that takes care of applying it only when the information is truthful.[28] Here, it is the operation that takes care of adding the implication's consequent only when the agent knows explicitly both the implication and its antecedent, thus simplifying the semantic interpretation of $[{}^{\xi}\hookrightarrow_{\chi}]$.

Basic properties. Intuitively, after the truth set of a given formula is added to the neighbourhood, the agent would acknowledge the formula as true. However, Moorean phenomena also occurs in these circumstances. Thus, after a formula's truth set has been added to the neighbourhood, the agent might not acknowledge such formula as true, even if it is the consequent of an explicitly known implication whose antecedent is also explicitly known.

FACT 3

$\not\Vdash (\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi) \rightarrow [{}^{\xi}\hookrightarrow_{\chi}]\,\mathrm{A}^{\mathrm{t}}\,\chi$.

PROOF. Take the atoms $\{p,q\}$ and the *ANM* $M = \langle\{w_1,w_2,w_3\},N,V,\{p,q\}\rangle$, with $V(p) = \{w_1\}, V(q) = \{w_1,w_2\}$ and

$$N(w_1) = \{\{w_1,w_2\},\{w_1,w_3\}\}, \qquad N(w_2) = \{\{w_1\}\}, \qquad N(w_3) = \{\{w_1\}\}.$$

Take the implication $q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)$. From $\mathrm{at}(q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)) \subseteq \{p,q\}$ it follows that $[\![\mathrm{A}^{\mathrm{o}}(q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)) \wedge \mathrm{A}^{\mathrm{o}}\,q]\!]^{M} = \{w_1,w_2,w_3\}$; moreover, $[\![q]\!]^{M} = \{w_1,w_2\}$ and

$$\begin{aligned}[\![q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)]\!]^{M} &= [\![\neg q]\!]^{M} \cup [\![p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p]\!]^{M}\\ &= [\![\neg q]\!]^{M} \cup ([\![p]\!]^{M} \cap [\![\neg\,\mathrm{A}^{\mathrm{t}}p]\!]^{M})\\ &= \{w_3\} \cup (\{w_1\} \cap \{w_1\})\\ &= \{w_1,w_3\},\end{aligned}$$

imply together that $[\![\mathrm{A}^{\mathrm{t}}(q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)) \wedge \mathrm{A}^{\mathrm{t}}\,q]\!]^{M} = \{w_1\}$. Therefore, it follows that $[\![\mathrm{K}_{Ex}(q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)) \wedge \mathrm{K}_{Ex}\,q]\!]^{M} = \{w_1,w_3\}$.

Now, the model after a modus ponens with $q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)$ is such that

$$N^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}(w_1) = \{\{w_1,w_2\},\{w_1,w_3\},\{w_1\}\},$$

$$N^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}(w_2) = N^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}(w_3) = \{\{w_1\}\}.$$

Thus, $[\![p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p]\!]^{M^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}} = [\![p]\!]^{M^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}} \cap [\![\neg\,\mathrm{A}^{\mathrm{t}}p]\!]^{M^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}} = \{w_1\} \cap \varnothing = \varnothing$, so $[\![\mathrm{A}^{\mathrm{t}}(p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)]\!]^{M^{q\hookrightarrow_{p\wedge\neg\mathrm{A}^{\mathrm{t}}p}}} = [\![[{}^{q}\hookrightarrow_{p\wedge\neg\,\mathrm{A}^{\mathrm{t}}p}]\,\mathrm{A}^{\mathrm{t}}(p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)]\!]^{M} = \varnothing$.

Putting the two pieces together,

$$w_1 \notin [\![\big(\mathrm{K}_{Ex}(q \rightarrow (p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)) \wedge \mathrm{K}_{Ex}\,q\big) \rightarrow [{}^{q}\hookrightarrow_{p\wedge\neg\,\mathrm{A}^{\mathrm{t}}p}]\,\mathrm{A}^{\mathrm{t}}(p \wedge \neg\,\mathrm{A}^{\mathrm{t}}p)]\!]^{M}.$$

□

Still, the property holds for those implications whose consequent's truth set is not affected by the operation (this includes all propositional formulas).

[28] For example, the case of a *public announcement* [29, 47], whose precondition is for the announced formula to be true.

PROPOSITION 2

$$\Vdash (\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi) \rightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\mathrm{A}^{\mathrm{t}}\,\chi \qquad \text{for } \chi \in \mathcal{L}_{[\hookrightarrow]} \text{ s.t. } \Vdash \chi \leftrightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\chi.$$

PROOF. Take $M = \langle W, N, V, A \rangle$ and $w \in W$ s.t. $w \in [\![(\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi]\!]^{M}$. Because of the latter, $M^{\xi \hookrightarrow \chi}$ is such that $[\![\chi]\!]^{M} \in N^{\xi \hookrightarrow \chi}(w)$. Then, by the required equivalence, $[\![[{}^{\xi}\!\hookrightarrow_{\chi}]\,\chi]\!]^{M} \in N^{\xi \hookrightarrow \chi}(w)$, i.e. $[\![\chi]\!]^{M^{\xi \hookrightarrow \chi}} \in N^{\xi \hookrightarrow \chi}(w)$. Thus, $w \in [\![\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M^{\xi \hookrightarrow \chi}}$, i.e. $w \in [\![[{}^{\xi}\!\hookrightarrow_{\chi}]\,\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M}$, as required. □

Its effect on explicit knowledge. From Fact 3, it is clear that

- $\not\Vdash (\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi) \rightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\mathrm{K}_{Ex}\,\chi.$

However, even though the modus ponens operation does not guarantee *awareness that*, it does guarantee *awareness of*,

$$\Vdash (\mathrm{K}_{Ex}(\xi \rightarrow \chi) \wedge \mathrm{K}_{Ex}\,\xi) \rightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\mathrm{A}^{\mathrm{o}}\,\chi,$$

the reason being that, knowing the implication explicitly, the agent is aware of it, and thus also aware of the implication's consequent, a fact that is not affected by the operation (the awareness set does not change). This, together with Proposition 2, yields the following *dynamic* version of the famous *K* axiom:

$$\Vdash \mathrm{K}_{Ex}(\xi \rightarrow \chi) \rightarrow (\mathrm{K}_{Ex}\,\xi \rightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\mathrm{K}_{Ex}\,\chi) \qquad \text{for } \chi \in \mathcal{L}_{[\hookrightarrow]} \text{ s.t. } \Vdash \chi \leftrightarrow [{}^{\xi}\!\hookrightarrow_{\chi}]\,\chi.$$

Observation. The actions studied so far, changes in *awareness of* and two flavours of deductive inference (the idealized full deductive closure described by $[*]$, and its just provided more realistic stepwise version), are typically understood as 'internal' actions; what the agent entertains changes because her focus might change, and what she has accepted as true changes as a consequence of her own self-reflection. But there are also 'external' epistemic actions that reflect the way the agent interacts with her environment. A paradigmatic one is the act of receiving further information from an external source, what in *DEL* has been called a *public announcement* [29, 47]. The focus of this manuscript has been the single-agent scenario, so here this action can be better called an act of *observation*.

Intuitively, by observing (or receiving, from a 100% reliable source, the information) that a given χ is indeed the case, the agent becomes both aware of it (she entertains it) and aware that it is the case.

DEFINITION 7 (Observation operation).
Let $M = \langle W, N, V, A \rangle$ be an *ANM*; let χ be a formula in $\mathcal{L}$. The model $M^{!\chi} = \langle W, N^{!\chi}, V, A^{!\chi} \rangle$ differs from M on the neighbourhood function (extending that of M with χ's truth set at M, when appropriate) and on the atomic awareness set (extending that of M with all atoms occurring in χ, as in the *becoming aware of* operation):

$$N^{!\chi}(w) := \begin{cases} N(w) \cup \{[\![\chi]\!]^{M}\} & \text{if } w \in [\![\chi]\!]^{M} \\ N(w) & \text{otherwise} \end{cases}, \qquad A^{!\chi} := A \cup \mathrm{at}(\chi).$$

Note: $[\![\chi]\!]^{M}$ is added only to the neighbourhood of those worlds in which χ holds. This is the action's precondition; for χ to be observed, it has to be the case.

For the syntax, $\mathcal{L}_{[!\chi]}$ extends $\mathcal{L}$ with an *observation* modality $[!\,\chi]$ for each formula χ. Formulas of the form $[!\,\chi]\,\varphi$ are read as '*φ is the case after the agent observes χ*',[29] and are semantically interpreted as

$$[\![[!\,\chi]\,\varphi]\!]^{M} = [\![\varphi]\!]^{M^{!\chi}}\ .$$

Basic properties. The operation's 'sanity check' is a twofold task: one needs to check that, after observing χ, the agent not only becomes aware of χ, but also acknowledges that the formula is the case. The first part is straightforward, as the operation adds all atoms in χ to the agent's awareness set, and thus has, in this respect, the same effect as the *becoming aware of* operation (Definition 3):

$$\Vdash [!\,\chi]\,\mathrm{A}^{\mathrm{o}}\,\chi.$$

However, the second requirement does not hold.

FACT 4
$\nVdash [!\,\chi]\,\mathrm{A}^{\mathrm{t}}\,\chi.$

PROOF. As in Fact 2, take $\neg\,\mathrm{A}^{\mathrm{o}}\,p$ and the model $M = \langle W = \{w\}, N, V, \varnothing\rangle$ over the set of atoms $\{p\}$, with the neighbourhood function given by $N(w) = \{W\}$. As before, $[\![\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{M} = W$ (so $w \in [\![\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{M}$). But then, the model $M^{!\neg\mathbf{A}^{\mathbf{o}}p}$ that results from the observation of $\neg\,\mathrm{A}^{\mathrm{o}}\,p$ is such that $N^{!\neg\mathbf{A}^{\mathbf{o}}p} = \{W\}$ and $A^{!\neg\mathbf{A}^{\mathbf{o}}p} = \{p\}$. Then, $[\![\mathrm{A}^{\mathrm{o}}\,p]\!]^{M^{!\neg\mathbf{A}^{\mathbf{o}}p}} = W$ (from the second) and therefore $[\![\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{A^{!\neg\mathbf{A}^{\mathbf{o}}p}} = \varnothing$; hence,

$$\begin{aligned}
[\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{M^{!\neg\mathbf{A}^{\mathbf{o}}p}} &= \{u \in W \mid [\![\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{M^{!\neg\mathbf{A}^{\mathbf{o}}p}} \in N^{!\neg\mathbf{A}^{\mathbf{o}}p}(u)\}\\
&= \left\{u \in W \mid \varnothing \in N^{!\neg\mathbf{A}^{\mathbf{o}}p}(u)\right\}\\
&= \varnothing.
\end{aligned}$$

Therefore, $[\![[!\,\neg\,\mathrm{A}^{\mathrm{o}}\,p]\,\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{o}}\,p]\!]^{M} = \varnothing$; at world w in the model that results after the agent observes $\neg\,\mathrm{A}^{\mathrm{o}}\,p$, the agent has *not* acknowledged $\neg\,\mathrm{A}^{\mathrm{o}}\,p$. □

Just as before, this instance of Moorean phenomena should not be taken as a proof that the provided definition of the observation operation is incorrect. For those formulas χ whose truth set is not affected by the operation (which, again, includes all purely propositional formulas), the expected property holds.

PROPOSITION 3

$$\Vdash \chi \rightarrow [!\,\chi]\,\mathrm{A}^{\mathrm{t}}\,\chi \qquad\qquad \text{for } \chi \in \mathcal{L}_{[!\chi]} \text{ such that } \Vdash \chi \leftrightarrow [!\,\chi]\,\chi.$$

PROOF. Let $M = \langle W, N, V, A\rangle$ be an *ANM*; take $w \in W$ with $w \in [\![\chi]\!]^{M}$. Because of the latter, $M^{!\chi}$ is such that $[\![\chi]\!]^{M} \in N^{!\chi}(w)$. Then, by the required equivalence, $[\![[!\,\chi]\,\chi]\!]^{M} \in N^{!\chi}(w)$, i.e. $[\![\chi]\!]^{M^{!\chi}} \in N^{!\chi}(w)$. Thus, $w \in [\![\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M^{!\chi}}$, i.e. $w \in [\![[!\,\chi]\,\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M}$. □

Its effect on explicit knowledge. From Fact 4, it follows that

- $\nVdash [!\,\chi]\,\mathrm{K}_{Ex}\,\chi.$

[29] Define $\mathrm{at}([!\,\chi]\,\varphi) := \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$.

However, from Proposition 3 and the way $[!\,\chi]$ affects what the agent entertains,

$$\Vdash \chi \rightarrow [!\,\chi]\,\mathrm{K}_{Ex}\,\chi \qquad \text{for all } \chi \in \mathcal{L}_{[!\chi]} \text{ such that } \Vdash \chi \leftrightarrow [!\,\chi]\,\chi .$$

Forgetting. Finally, just as the agent can accept new information as true (via internal reasoning [modus ponens] or external interaction [observation]), she can also ‘forget’ some of it. This action is the *awareness that* counterpart of the act of *becoming unaware of*; what the agent entertains remains the same, but afterwards the agent might not be sure about the given formula’s truth-value.[30]

DEFINITION 8 (*Forgetting* operation).
Let $M = \langle W, N, V, A\rangle$ be an *ANM*-model; let χ be a formula in $\mathcal{L}$. The model $M^{\backslash\chi} = \langle W, N^{\backslash\chi}, V, A\rangle$ differs from M only in its neighbourhood function, defined for $w \in W$ as

$$N^{\backslash\chi}(w) = N(w) \setminus \{[\![\chi]\!]^M\}.$$

For the language, $\mathcal{L}_{[\backslash\chi]}$ extends with a modality for the *forgetting* operation; if χ and φ are formulas in the resulting extended language, then so is $[\backslash\chi]\,\varphi$, read as ‘*$\varphi$ holds after the agent forgets χ*’,[31] and semantically interpreted as

$$[\![[\backslash\chi]\,\varphi]\!]^M = [\![\varphi]\!]^{M^{\backslash\chi}} .$$

Basic properties. Intuitively, after the truth set of a given formula is removed from the neighbourhood, the agent would not acknowledge the formula as true. However, just as in the previous cases, Moorean phenomena occurs; the agent might accept a formula as true, remove its truth set from the neighbourhood, and yet still consider it true afterwards.

FACT 5
$\not\Vdash \mathrm{A}^{\mathrm{t}}\,\chi \rightarrow [\backslash\chi]\,\neg\,\mathrm{A}^{\mathrm{t}}\,\chi$.

PROOF. Take $\mathrm{P} = \{p\}$ and $M = \langle W = \{w_1, w_2\}, N, V, \varnothing\rangle$, with $V(p) = \{w_2\}$ and

$$N(w_1) = \{\{w_2\}, W\}, \qquad N(w_2) = \{\,\}.$$

Note how $[\![\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^M = W \setminus [\![\mathrm{A}^{\mathrm{t}} p]\!]^M = W \setminus \{w_1\} = \{w_2\}$; hence, $[\![\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^M \in N(w_1)$ and thus $w_1 \in [\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^M$. However, $M^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}$ is such that

$$N^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}(w_1) = \{W\}, \qquad N^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}(w_2) = \{\,\}.$$

Thus, $[\![\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^{M^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}} = W \setminus [\![\mathrm{A}^{\mathrm{t}} p]\!]^{M^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}} = W$; hence, $[\![\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^{M^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}} \in N^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}(w_1)$ so $w_1 \in [\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^{M^{\backslash\neg\mathbf{A}^{\mathbf{t}}\boldsymbol{p}}}$, i.e. $w_1 \in [\![[\backslash\neg\,\mathrm{A}^{\mathrm{t}} p]\,\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^M$.

Therefore, $w_1 \in [\![\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}} p \wedge [\backslash\neg\,\mathrm{A}^{\mathrm{t}} p]\,\mathrm{A}^{\mathrm{t}}\,\neg\,\mathrm{A}^{\mathrm{t}} p]\!]^M$. □

As before, the expected property holds for those implications whose consequent’s truth set is not affected by the operation.

PROPOSITION 4

$$\Vdash [\backslash\chi]\,\neg\,\mathrm{A}^{\mathrm{t}}\,\chi \qquad \text{for } \chi \in \mathcal{L}_{[\backslash]} \text{ s.t. } \Vdash \chi \leftrightarrow [\backslash\chi]\,\chi$$

[30] Cf. the relational-model-based *forgetting* operations of [17, 25].
[31] Define $\mathrm{at}([\backslash\chi]\,\varphi) := \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$.

PROOF. Take $M = \langle W, N, V, A\rangle$ and $w \in W$. From $N^{\backslash\chi}$'s definition, $[\![\chi]\!]^M \notin N^{\backslash\chi}(w)$; thus, from the equivalence, $[\![[\backslash\chi]\,\chi]\!]^M \notin N^{\backslash\chi}(w)$, i.e. $[\![\chi]\!]^{M^{\backslash\chi}} \notin N^{\backslash\chi}(w)$, and therefore $w \notin [\![\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M^{\backslash\chi}}$, so $w \in [\![\neg\,\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M^{\backslash\chi}}$ and thus $w \in [\![[\backslash\chi]\,\neg\,\mathrm{A}^{\mathrm{t}}\,\chi]\!]^{M}$. □

Its effect on explicit knowledge. From Fact 5, it follows that

- $\not\Vdash [\backslash\chi]\,\neg\,\mathrm{K}_{Ex}\,\chi$.

However, from Proposition 4,

$$\Vdash [\backslash\chi]\,\neg\,\mathrm{K}_{Ex}\,\chi \qquad \text{for } \chi \in \mathcal{L}_{[\backslash]} \text{ s.t. } \Vdash \chi \leftrightarrow [\backslash\chi]\,\chi.$$

5 Conclusions and future work

This paper proposes a formal setting defining a notion of *explicit knowledge* based on two forms of awareness: *awareness of* and *awareness that*. The resulting notion of knowledge does not suffer from the idealizations that other notions of knowledge obtain under other semantic structures, as it separates the mere fact of entertaining some information (being *aware of* φ) from the acknowledgement that the information is indeed the case (being *aware that* φ holds). All these three notions have been defined formally, and their most important properties have been discussed.

The proposed framework has other appealing features. First, it allows a natural definition of the notion of *implicit knowledge*, understood as the closure under logical consequence of its *explicit* counterpart, in terms of the well-known relationship between neighbourhood and relational models. Second, it also describes other epistemic notions that arise from combining the two mentioned forms of awareness, as what the agent has acknowledged as true but is not currently entertaining (informally, *disassociated* knowledge), or what she is not currently entertaining, and yet she can deduce from what she has acknowledged as true (*currently unreachable* knowledge). Third, it allows the representation of different epistemic actions, some of them affecting what the agent entertains, and some others affecting what she has accepted as true. These actions have been defined, and it has been discussed the way the agent's explicit knowledge is affected by each one of them and the way the agent's explicit knowledge is affected by each one of them has been discussed.

There are several directions in which this proposal can be extended. On the technical side, the most important is a sound and complete axiom system axiomatizing not only the basic framework (Section 2) but also its dynamics extensions (Section 4); this system is under development and has been left, for space reasons, for a companion manuscript.[32] On the conceptual side, an important next step is the move to the multi-agent setting, which will allow the representation of the information that 'real' agents have about the information of each other. Moreover, when several agents are involved, one can look at the public and private versions of the actions presented here, which will allow the agents to change their epistemic state in a way that might not be observed by everyone. Finally, an appealing further direction is the extension to a setting in which the modelled agent has not only knowledge (in a more proper sense, probably satisfying the *truth axiom* $\mathrm{K}\,\varphi \rightarrow \varphi$), but also beliefs. This would allow the discussion of the fine-grained versions of other epistemic actions, starting with *belief revision*, but also including forms of non-deductive inference (cf. [60, 61]).

[32] Still, the most important properties of the main concepts have been discussed (Section 3).

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Received 31 January 2019