Some Operations Preserving Primitivity of Words

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Abstract

We investigate some operations where essentially, from a given word $w$, the word $ww'$ is constructed where $w'$ is a modified copy of $w$ or a modified mirror image of $w$. We study whether $ww'$ is a primitive word provided that $w$ is primitive. For instance, we determine all cases with an edit distance of $w$ and $w'$ at most 2 such that the primitivity of $w$ implies the primitivity of $ww'$. The operations are chosen in such a way that in the case of a two-letter alphabet, all primitive words of length $\leq 11$ can be obtained from single letters.

1 Introduction

A word $w$ over an alphabet $V$ is said to be a primitive word if and only if there is no word $u \in \Sigma^+$ with $w = u^n$ for some natural number $n > 1$. The set of all primitive words over $V$ is denoted by $Q_V$. There are a lot of papers on relations of $Q_V$ to other language families as the families of the Chomsky hierarchy (e.g. in [4] and [17], it has been shown that $Q_V$ is neither a deterministic nor an unambiguous context-free language, in [8] relations to regular languages are given), Marcus contextual grammars (see [6]), to (poly-)slender languages (see [5]) and some languages and language families related to codes.
(see e.g. [19]). Moreover, there are papers on combinatorial properties of primitive words and of the sets $Q_V$; we refer to [2], [1], [9].

However, there is only a small number of results concerning the closure of $Q_V$ under operations. There are some papers where it was investigated whether the application of homomorphisms to primitive words leads to primitive words in all cases or leads to primitive words with a finite number of exceptions or to non-primitive words in all cases; we refer to [13], [14], [15], [10]. In [18] homomorphisms are studied which preserve the property to be a Lyndon word or to be border-free (a word $w$ is a Lyndon word if and only if any non-empty proper suffix of $w$ is greater than $w$ with respect to the lexicographic order; it is border-free if there is no non-empty word which is a proper prefix as well as a proper suffix of $w$); it is shown that such homomorphisms preserve primitivity, too. Substitutions form another operation which was investigated with respect to preservation of primitivity. There were substitutions of very short subwords in the focus, especially point mutations (deletions, insertions and substitutions of one letter) were studied. We refer to [16] for details. A further study in this direction concerns insertions (see [11]).

Obviously, there is a large variety of operations from which one can expect that $Q_V$ is closed under them (since the portion of primitive words is very high). In this paper we consider some operations where essentially, from a given word $w$, the word $ww'$ is constructed where $w'$ is a modified copy of $w$ or a modified mirror image of $w$. The modifications are of such a form that the edit distance of $w$ and $w'$ is very small or very large (i.e., it is very near to the length of $w$).

We have two reasons for this investigation. The first one is of combinatorial nature. Obviously, $ww'$ is not primitive for all $w$. We are interested in conditions for changes of the second copy $w$ to $w'$ such that $ww'$ is primitive for all $w$. Especially, how many changes or deletions or insertions of letters are necessary and how many such operations are possible. For example, we shall determine all possible transformation where the edit distance of $w$ and $w'$ is at most two and primitivity is preserved. The second reason comes from the theory of dynamical systems. In the paper [7] a dynamical system based on regular languages has been proposed. The regular languages are essentially described by primitive words. Since in dynamical systems one needs mutations in order to develop the system, one is interested in devices which describe primitive words and allow mutations. Here the use of operations which preserve primitivity is of interest. Then a primitive word can be given as a sequence of operations; and a mutation is the replacement of one operation by another one or a deletion or insertion of an operation in the sequence. This ensures primitivity of the word obtained from the mutated sequence of operations. Obviously, it is not necessary to generate all primitive words, however, the set of generated primitive words should contain a good approximation of any primitive word where the quality of approximations is determined by the dynamic system (especially its fitness
function). We have chosen the operations under which $Q_V$ is closed in such a way that, if the underlying alphabet $V$ consists of two letters, then by the operations we can generate all primitive words of length $\leq 11$ (as can be shown by computer calculations) and a sufficient large amount of primitive words of the length up to twenty.

Thus this paper can also be considered as a continuation of the investigations of devices generating only primitive words (see e.g. [3]).

The paper is organized as follows. In Section 2, we present and recall some notations and some results on primitive words which are used in the sequel. In Section 3, we introduce some operations where we first construct $ww$ and perform then some small modifications of the second copy yielding $ww'$. We prove that all operations where the edit distance of $w$ and $w'$ is 1 preserve primitivity. An analogous result is shown for the edit distance 2 if at least one change of a letter is used. In Section 4, we consider analogous operations as in Section 2, but start from $ww^R$ and modify $w^R$. In Section 5 we consider $ww'$ where $w'$ is obtained from $w$ or $w^R$ by a drastic change, i.e., the Hamming distance of $w'$ and $w$ or $w^R$ is almost the length of $w$. Moreover, we give some further operations where the length is almost doubled and primitivity is preserved.

2 Some Notation and Facts

By $\#(A)$ we denote the cardinality of a set $A$.

For a given alphabet $V$, we denote by $V^*$ and $V^+$ the set of all and all non-empty words over $V$, respectively. The empty word is designated by $\lambda$. Given a word $w \in V^*$ and $x \in V$, we denote its length by $|w|$ and the number of occurrences of $x$ in $w$ by $\#_x(w)$. For a word $w = x_1x_2\ldots x_n \in V^+$ with $x_i \in V$ for $1 \leq i \leq n$, we define the mirror image $w^R$ by $w^R = x_nx_{n-1}\ldots x_1$. Given two words $w = x_1x_2\ldots x_n \in V^+$ and $w' = y_1y_2\ldots y_n \in V^+$ with $x_i, y_i \in V$ for $1 \leq i \leq n$, the Hamming distance $d(w, w')$ is defined by $d(w, w') = \#(\{i \mid x_i \neq y_i\})$ and the edit distance $ed(w, w')$ of $w$ and $w'$ is the minimal number of changes, deletions and insertions of letters in order to transform $w$ into $w'$.

Throughout the paper we assume that $V$ has at least two elements.

A word $w \in V^+$ is said to be a primitive word if and only if there is no word $u \in V^+$ such that $w = u^n$ for some natural number $n > 1$. By $Q_V$ we denote the set of all primitive words over $V$. If $V$ is understood from the context we omit the index $V$ and write simply $Q$.

We recall three facts (see [12], [19], [1]) which will be used in the sequel.

**Lemma 1** For any words $v, v' \in V^*$, $vv' \in Q$ if and only if $v'v \in Q$.

**Lemma 2** For two non-empty words $u$ and $v$, $uv = vu$ if and only if there is
a word $z$ such that $u = z^n$ and $v = z^m$ for some natural numbers $n$ and $m$. □

**Lemma 3** In a free monoid $V^*$, the equation $a^m b^n = c^p$, where $a, b, c \in V^*$ and $m, n, p \geq 2$, has only trivial solutions, where $a$, $b$ and $c$ are powers of some word in $V^*$. □

**Lemma 4** (Fine-Wilf Theorem) Let $u, v \in V^+$ and $n, m \geq 2$. If $u^n$ and $v^m$ have a common prefix of length at least $|u| + |v| - \gcd(|u|, |v|)$, then $u$ and $v$ are powers of the same primitive word. □

The following statement holds trivially.

**Lemma 5** If $w \in Q$, then also $w^R \in Q$. □

Lemmas 1 and 5 can be interpreted as follows: If we apply a cyclic shift or the mirror image to a primitive word, then we obtain a primitive word, again. Thus cyclic shifts and reversal are operations which preserve primitivity.

**Lemma 6** For any $x \in V$, $y \in V$ and $z \in V^*$, if $xz = zy$, then $x = y$.

*Proof*. If $z = \lambda$, then $x = y$ immediately. If $z = a_1 a_2 \ldots a_n$ with $a_i \in V$ for $1 \leq i \leq n$, then $x = a_1, a_1 = a_2, a_2 = a_3, \ldots a_{n-1} = a_n, a_n = y$ and consequently $x = y$. □

In the sequel we shall use the following notation. If $w = w_1 w_2 \ldots w_r = z_1 z_2 \ldots z_s$ for some words $w_1, \ldots w_r, z_1, \ldots, z_s \in V^*$ such that $|w_1 w_2 \ldots w_i| = |z_1 z_2 \ldots z_j|$ for some $i$ and $j$, we write

$$w_1 w_2 \ldots w_i | w_{i+1} w_{i+2} \ldots w_r = z_1 z_2 \ldots z_j | z_{j+1} z_{j+2} \ldots z_s,$$

i.e., by the symbol $|$ we mark a certain position in the word. (Some authors write $(w, w') = (z, z')$ instead of $w|w' = z|z'$.) Mostly, $|$ will mark the middle of a word of even length, or it will be put after the $m$-th letter if the word has odd length $2m - 1$.

### 3 Operations with an Almost Duplication

Obviously, the word $ww$ obtained from $w$ by a duplication leads from any word $w$ to a non-primitive word. In order to obtain primitive words from a primitive word $w$ one has to perform some changes in the second occurrence of $w$, i.e., one has to consider words of the form $ww'$ where $w'$ differs only slightly from $w$. In most cases the edit distance of $w$ and $w'$ will be at most 2, and thus $ww'$ can be considered as an almost duplication of $w$.

We start with the case where we only change some letters to obtain $w'$ from $w$. 

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Theorem 7 

i) Let \( w \) be a primitive word of some length \( n \) and \( w' \) an arbitrary word of length \( n \) such that the Hamming distance \( d(w, w') \) is a power of 2, then \( ww' \) is primitive, too.

ii) If \( d \) is not a power of 2, then there are a primitive word \( w \) and a word \( w' \) with \( d(w, w') = d \) such that \( ww' \) is not a primitive word.

Proof. 

i) Obviously, \(|ww'|\) is even. Let us suppose \( ww' \notin Q \), that is, there exists \( p \in \mathbb{N} \) and \( v \in V^+ \) of length at least 2 such that \( ww' = v^p \).

If \( p \) is even, then \( w = w' = v^\frac{p}{2} \) since \(|v| = |w'|\). Thus \( d(w, w') = 0 \) which contradicts the assumption on the Hamming distance of \( w \) and \( w' \).

If \( p \) is odd, i.e., \( p = 2m + 1 \) for some \( m \geq 1 \), then \(|v|\) is even (since otherwise \(|v|p = |ww'| \) would be odd). Thus there are words \( v' \) and \( v'' \) of length \( \frac{|v|}{2} \) such that \( v = v'v'' \). Then we get \( w = v^m v' = (v'v'')^m v' \) and \( w' = v'' v^m = v'' (v'v'')^m \).

Then \( d(w, w') = (2m + 1)d(v', v'') \). Since \( 2m + 1 \) is an odd number, \( d(w, w') \) is not a power of 2 in contrast to our supposition.

ii) Let \( d \) be not a power of 2. Then there is an odd number \( q > 1 \) and a number \( p \) such that \( d = qp \). Let \( q = 2m + 1 \) for some \( m \geq 1 \). We now set

\[
\begin{align*}
v' &= 10^p, \\
v'' &= 11^p, \\
w &= (v'v'')^m v', \quad \text{and} \\
w' &= (v''v')^m v''.
\end{align*}
\]

Obviously, \( w \) is primitive, \( d(w, w') = (2m + 1)d(v', v'') = (2m + 1)p = qp = d \) and \( ww' = (v'v'')^{2m+1} \notin Q \). \( \square \)

By part ii) of the preceding theorem, if \( w \) is a primitive word and \( d(w, w') \) is not a power of 2, in general, \( ww' \) is not a primitive word. However, if we require that the changes occur in special positions it is possible to obtain preservation of primitivity. As an example we give the following operation.

Definition 8  

For any odd natural numbers \( n \geq 3 \), any alphabet \( V \), and any mapping \( h : V \to V \) with \( h(a) \neq a \) for all \( a \in V \), we define the operation \( O_{n,h} : V^n \to V^{2n} \) by

\[
O_{n,h}(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n h(x_1) x_2 \ldots x_{i-1} h(x_i) x_{i+1} \ldots x_{n-1} h(x_n)
\]

where \( i = \frac{n+1}{2} \).

Theorem 9  

For any odd natural number \( n \neq 5 \), any primitive word \( q \) of length \( n \), and any mapping \( h : V \to V \) with \( h(a) \neq a \) for all \( a \in V \), \( O_{n,h}(q) \) is a primitive word.

Proof.  

Let \( w = x_1 x_2 \ldots x_n \) with \( x_j \in V \) for \( 1 \leq j \leq n \) and \( i = \frac{n+1}{2} \). Then \( O_{n,h}(x_1 x_2 \ldots x_n) \) has an even length.

Let us suppose that \( O_{n,h}(w) \notin Q \), that is, there exist a \( p \geq 2 \) and \( v \in Q \) such that \( O_{n,h} = v^p \).

If \( p \) is even then

\[
v^\frac{p}{2} = x_1 x_2 \ldots x_{n-1} x_n = h(x_1) x_2 x_3 \ldots x_{i-1} h(x_i) x_{i+1} x_{i+2} \ldots x_{n-1} h(x_n).
\]
Thus $x_i = h(x_i)$, which is a contradiction.

Thus $p$ is odd, say $p = 2m + 1$ for some $m \geq 1$. As above there are words $v$, $v_1$ and $v_2$ such that $v = v_1v_2$ and $|v_1| = |v_2|$ and

$$x_1 \ldots x_{n-1}x_n|h(x_1)x_2 \ldots x_{i-1}h(x_i)x_{i+1} \ldots x_{n-1}h(x_n) = (v_1v_2)^mv_1v_2(v_1v_2)^m.$$  

Since $v_1$ starts with $x_1$ (first occurrence) and ends with $x_n$ (last occurrence in the first part), $v_1 = x_1v'_1x_n$ and analogously, $v_2 = h(x_1)v'_2h(x_n)$. Therefore we have that $O_{n,h}(w)$ has the form

$$(x_1v'_1x_nh(x_1)v'_2h(x_n))^m x_1v'_1x_nh(x_1)v'_2h(x_n)(x_1v'_1x_nh(x_1)v'_2h(x_n))^m.$$  

Since the letters $x_i$ and $x_n$ do not occur in the first occurrence of $v$, by the definition of $O_{n,h}$, the last letter of the first occurrence of $v_1$ (in the first part of the word) and last letter of the first occurrence of $v_2$ in the second part coincide, i.e., $x_n = h(x_n)$ which is a contradiction. \qed

The supposition $n \geq 5$ in Theorem 9 is necessary since the statement does not hold for $n = 3$ as can be seen from the following example. Let $q = aba \in Q$. Then $O_{3,h}(q) = ababab = (ab)^3 \notin Q$.

We now discuss some operations where the edit distance of $w$ to $w'$ is at most 2 and at least one deletion or one insertion is performed to obtain $w'$; more precisely, we consider

(a) the deletion of an arbitrary letter,
(b) the deletion of an arbitrary letter and the change of an arbitrary remaining letter,
(c) the insertion of an arbitrary letter,
(d) the insertion of an arbitrary letter and the change of an arbitrary letter of $w$.

We now give the formal definition of these operations.

**Definition 10** For any natural numbers $n, i, j, i'$ with $1 \leq i \leq n$, $0 \leq i' \leq n$, $1 \leq j \leq n$ and $i \neq j$, letters $x, y, z \in V$ with $x \neq y$, and a word $w = x_1x_2 \ldots x_n$, $x_i \in V$, of length $n$, we define the following operations

$$D_{n,i}, D_{n,i,j,x,y} : V^n \to V^{2n-1} \text{ and } I_{n,i',x,z}, I_{n,i',z,j,x,y} : V^n \to V^{2n+1}$$  

by
\[ D_{n,i}(x_1x_2\ldots x_n) = x_1x_2\ldots x_n x_1x_2\ldots x_{i-1}x_{i+1}x_{i+2}\ldots x_n, \]
\[ D_{n,i,j,x,y}(x_1\ldots x_n) = \begin{cases} x_1\ldots x_n x_1\ldots x_{i-1}x_{i+1}\ldots x_{j-1}yx_{j+1}\ldots x_n x_j = x, i < j \\ undefined & otherwise \end{cases}, \]
\[ I_{n,i',z}(x_1x_2\ldots x_n) = x_1x_2\ldots x_n x_1x_2\ldots x_{i'}zx_{i'+1}x_{i'+2}\ldots x_n, \]
\[ I_{n,i',z,j,x,y}(x_1\ldots x_n) = \begin{cases} x_1\ldots x_n x_1\ldots x_{i'}zx_{i'+1}\ldots x_{j-1}y x_{j+1}\ldots x_n x_j = x, i' < j \\ undefined & otherwise \end{cases}. \]

**Theorem 11** If \( n \geq 2, 1 \leq i \leq n, \) and \( q \) is a primitive word of length \( n \), then \( D_{n,i}(q) \in Q \) also holds.

**Proof.** Let \( q = uav \) for some \( u, v \in V^* \) and \( a \in V \). If \( |q| = 1 \), i.e., \( q = a \), then \( D_{n,i}(q) = a \in Q \).

If \( |q| \geq 2 \), then \( D_{|q|,|u|+1}(q) = uavuv \). Let us suppose that \( uavuv \notin Q \). Then \((vu)^2a \notin Q \) by Lemma 1. Let \((vu)^2a \notin Q \) by Lemma 1. Let \((vu)^2a = z^m \) for some \( z \in V^+ \) and some \( m \geq 2 \). Thus \((vu)^2\) is a common prefix of \((vu)^2\) and \( z^m \). Since
\[ |uv| + |z| = |uv| + \frac{2|uv| + 1}{m} \leq |vu| + \frac{2|uv| + 1}{2} < 2|vu| + 1, \]
we have
\[ |vu| + |z| - gcd(|vu|, |z|) < (2|vu| + 1) - 1 = 2|vu| = |(vu)^2|. \]

By Lemma 4, we obtain \((vu)^2 = w^k\) and \((vu)^2a = z^m = w^l\) for some \( w \in V^+ \) and some numbers \( k \) and \( l \). Obviously, \( w = a \). Hence \( u \) and \( v \) are powers of \( a \) and thus \( q \) is a power of \( a \). This contradicts the primitivity of \( q \). \( \square \)

**Theorem 12** If \( w \in V^+ \) and \( D_{n,i,j,x,y}(w) \) is defined, then \( D_{n,i,j,x,y}(w) \in Q \) holds.

**Proof.** We first discuss \( D_{n,n,j,x,y} \). Let \( w = x_1x_2\ldots x_n \). Then
\[ D_{n,n,j,x,y}(w) = x_1x_2\ldots x_{j-1}xx_{j+1}x_{j+2}\ldots x_n x_1x_2\ldots x_{j-1}yx_{j+1}x_{j+2}\ldots x_{n-1}. \]

Let us assume that \( D_{n,n,j,x,y}(w) \notin Q \). Then there is a word \( v \in V^+ \) such that \( D_{n,n,j,x,y}(w) = v^p \) for some \( p \geq 2 \). Since \( D_{n,n,j,x,y}(w) \) has odd length, \( p \) and the length of \( v \) are odd numbers. Let \( p = 2m + 1 \) for some \( m \geq 1 \). Thus there are words \( v_1 \in V^+ \) and \( v_2 \in V^+ \) such that \( v = x_1v_1v_2, k - 1 = |v_1| = |v_2| \) and
\[ x_1x_2\ldots x_{j-1}xx_{j+1}x_{j+2}\ldots x_n x_1x_2\ldots x_{j-1}yx_{j+1}x_{j+2}\ldots x_{n-1} = v^m x_1v_1v_2v^m. \]
Then \(|v| = 2k - 1\). We set \(s = 2k - 1\). We distinguish some cases.

**Case 1.** Let \(1 \leq j \leq k - 1\). Then by definition of \(D_{n,n,j,x,y}\),

\[
x_1v_1 = x_1x_2 \ldots x_{j-1}xx_{j+1} \ldots x_{k-1}x_k = z_1xz_2x_k
\]

and

\[
v_2 = x_1x_2 \ldots x_{j-1}yx_{j+1} \ldots x_{k-1} = z_1yz_2.
\]

Thus, we get,

\[
v = z_1xz_2x_kz_1yz_2.
\]

If \(m \geq 2\), the first part of the word is

\[
z_1xz_2x_kz_1yz_2z_1xz_2x_kz_1yz_2y^{m-2}z_1xz_2x_k
\]

and that of the second part is

\[
z_1yz_2z_1xz_2x_kz_1yz_2z_1xz_2x_kz_1yz_2y^{m-2}
\]

and these two words differ in the \(|z_1xz_2x_kz_1yz_2z_1| + 1\)-st letter, which contradicts the definition of \(D_{n,n,j,x,y}\). If \(m = 1\), the first and second part are

\[
z_1xz_2x_kz_1yz_2z_1xz_2x_k \text{ and } z_1yz_2z_1xz_2x_kz_1yz_2,
\]

respectively, and we get a contradiction as above.

**Case 2.** Let \(j = k\). Then the \(k\)-th letter in the second part is \(y\). On the other hand, it is \(x_1\) since there starts the word \(v\). Thus \(x_1 = y\). This gives

\[
x_1v_1 = x_1x_2 \ldots x_{k-1}x_k = yzx, \quad v_2 = x_1x_2 \ldots x_{k-1} = yz \quad \text{and} \quad v = yzyyz
\]

with \(z = x_2x_3 \ldots x_{k-1}\). Then the first and second part are

\[
yyzxyyyzyyzv^{m-2}yzyx \quad \text{and} \quad yzyzzxyyzzyzvy^{m-2},
\]

respectively. We obtain \(zx = yz\) by looking on the words starting in the position \(|z| + 3\). Thus by Lemma 6, \(x = y\) in contrast to the definition of \(D_{n,n,j,x,y}\).

**Case 3.** Let \(k+1 \leq j \leq 2k-1\). Then \(v = x_1v_1v'_2xv''_2\). Moreover, \(|v'_2| = j - k - 1\). Furthermore, \(y\) stands in the \(j\)-th position of \(v'_2xv''_2x_1v_1\), i.e., \(x_1v_1 = x_1v'_1yv''_1\) with \(|v'_1| = j - k - 1\). Therefore \(v = x_1v'_1yv''_1v'_2xv''_2\) and \(|v'_1| = |v''_1| = |v''_2|\).

Then we get for the second part

\[
x_1v'_1yv''_1v'_2yv''_2x_1v'_1yv''_1v'_2xv''_2x_{2s-1}x_{2s} \ldots x_n
\]

by the definition of \(D_{n,n,j,x,y}\) and from the form

\[
v'_2xv''_2x_1v'_1yv''_1v'_2xv''_2v^{m-1}
\]
given by our assumption. Considering the words which start in the position
\((|x_1v_1'yv''_1| + 1)\) and in the position \(\left(|x_1v_1'yv''_1v_2'y| + 1\right)\), respectively, we see that
\(v'_1 = v'_2 = z\) and \(v''_1 = v''_2 = z'\). Looking on the subwords starting in the first
position and in the position \(|v'_1| + 2\), we get \(x_1z = zx\) and \(yz' = xx_1\). By
Lemma 6, \(x_1 = x\) and \(y = x_1\), which contradicts \(x \neq y\).

Case 4. Let \(j = hs + q\) for some \(h \geq 1\) and \(1 \leq q \leq k - 1\). Then \(x_j = x\) is the
\(q\)-th letter of \(v\). Thus \(v = v'_1xv''_1v_2\) with \(|v'_1| = q - 1\).

We now compute the position of \(y\) in \(v\). Since the second part starts with \(v_2\) of
length \(k - 1\) and \(hs + q = k - 1 + (h - 1)s + s + q - (k - 1) = k_1 + (h - 1)s + k + q,\) \(y\) is
the \((k + q)\)-th letter of \(v\). Therefore \(v = v'_1xv''_1v_2y\) with \(|v'_1| = |v''_1|\). Moreover,
\(|v'_1| = |v''_1| + 1\). Now we get easily the same situation as in Case 1; thus we get
(1) and (2) and a difference in the \((|z_1| + 1)\)-st position.

Case 5. Let \(j = hs + k\) for some \(h \geq 1\). Then \(x\) is the \(k\)-th letter of \(v\). We
compute the position of \(y\) in \(v\). Since the second part starts with \(v_2\) of length
\(k - 1\) and \(hs + k = k - 1 + hs + k - (k - 1),\) \(y\) is the first letter of \(v\). Therefore
we get \(v = yz\) as in Case 2, which leads to a contradiction.

Case 6. Let \(j = hs + q\) for some \(h \geq 1\) and \(k + 1 \leq q \leq 2k - 1\). Then
\(x_j = x\) is the \(q\)-th letter of \(v\). Thus \(v = x_1v_1v'_2xv''_2\) with \(|x_1v_1v'_2| = q - 1 \geq k\).
Furthermore, \(|v''_2| = 2k - 1 - q\).

We now compute the position of \(y\) in \(v\). Since the second part starts with \(v_2\) of
length \(k - 1\) and \(hs + q = k - 1 + hs + q - (k - 1),\) \(y\) is the \((q - k + 1)\)-st
letter of \(v\). Therefore

\[ v = x_1v_1'yv''_1v_2xv''_2 \] with \(|x_1v'_1| = q - k\).

Therefore \(|v'_1| = k - (q - k + 1) = 2k - 1 - q\). Hence \(|v''_1| = |v''_2|\) and consequently
\(|v'_1| = |v''_2|\). Therefore we have exactly the situation of Case 3, which leads to
contradiction.

Let us now consider \(i = 1,\) i.e., the operation \(D_{n,1,j,x,y}\). By the first part of this
proof

\[ D_{n,n,n-j+1,x,y}(w^R) = x_nx_{n-1} \ldots x_1x_nx_{n-1} \ldots x_{j+1}yx_{j-1}x_{j-2} \ldots x_2 \in Q, \]

by Lemma 5,

\[ x_2x_3 \ldots x_{j-1}yx_{j+1}x_{j+2} \ldots x_{n}x_1x_2 \ldots x_n \in Q, \]

and by Lemma 1

\[ x_1x_2 \ldots x_nx_2x_3 \ldots x_{j-1}yx_{j+1}x_{j+2} \ldots x_n = D_{n,1,j,x,y}(w) \in Q. \]

We now consider the case \(j < i\). We set

\[ \overline{w} = x_{i+1}x_{i+2} \ldots x_{n}x_1x_2 \ldots x_i. \]
Moreover, let \( x_j = x \). By the first part of this proof we get

\[
D_{n,n-i+j,x,y}(w) = x_{i+1} \ldots x_n x_1 \ldots x_i x_{i+1} \ldots x_n x_1 \ldots x_{j-1} y x_j+1 \ldots x_{i-1} \in Q.
\]

Hence, by Lemma 1

\[
x_1 \ldots x_i x_{i+1} \ldots x_n x_1 \ldots x_{j-1} y x_j+1 \ldots x_{i-1} x_{i+1} \ldots x_n = D_{n,i,j,x,y}(w) \in Q.
\]

If \( i < j \) we can prove that \( D_{n,i,j,x,y}(w) \in Q \) analogously to the case \( j < i \) using \( D_{n,1,j,x,y} \) instead of \( D_{n,n,j,x,y} \). \( \square \)

**Theorem 13** If \( q \) is a primitive word of length \( n \), \( 0 \leq i \leq n \) and \( z \in V \), then \( I_{n,i,z}(q) \in Q \).

**Proof.** Let \( q \) be a primitive word of length \( n \) and \( a \in V \). Let \( u \) be the prefix of \( q \) of length \( i \) and \( q = uv \). Then \( I_{n,i,a}(w) = uvuav \). If \( uvuav \not\in Q \), we can derive a contradiction as in the proof of Theorem 11. \( \square \)

**Theorem 14** If \( q \in Q \) and \( I_{n,i,z,j,x,y}(q) \) is defined, then \( I_{n,i,z,j,x,y}(q) \in Q \).

**Proof.** Let \( w = x_1 x_2 \ldots x_{j-1} x x_{j+1} x_{j+2} \ldots x_n \). Then

\[
I_{n,n,a,j,x,y} = x_1 x_2 \ldots x_n x_1 x_2 \ldots x_{j-1} y x_{j+1} x_{j+2} \ldots x_n a.
\]

If we assume that \( I_{n,n,a,j,x,y} \) is not in \( Q \), then

\[
x_1 \ldots x_{j-1} y x_{j+1} \ldots x_n a x_1 \ldots x_n = D_{n+1,n+1,j,y,x}(x_1 \ldots x_{j-1} y x_{j+1} \ldots x_n a) \not\in Q,
\]

which is a contradiction to Theorem 12. The general case can be obtained using Lemmas 1 and 5. \( \square \)

Let \( ww' \) be given with \( ed(w, w') = 1 \). Then \( w' \) is obtained by a change (i.e., \( d(w, w') = 1 = 2^0 \)), either by a deletion or by an insertion. By the Theorems 7, 11 and 13, \( ww' \) is in \( Q \) provided that \( w \in Q \). If \( ed(w, w') = 2 \) we have again \( ww' \in Q \) if two changes, or a deletion and a change, or a change and an insertion are performed (by Theorems 7, 12 and 14). In the remaining cases, in general, primitivity is not preserved. Performing two deletions we can get a non-primitive word, as can be seen from \( w = 110^p1 \) which results in \( 110^p1110^p1 \) and gives \( 110^p1110^p = (110^p)^2 \not\in Q \) if we delete the first and last letters of the second copy (note that the statement holds for any length \( n \geq 4 \) since it holds for any \( p \geq 1 \)). The same holds for two insertions; e.g., the duplication \( 10^p10^p \) of \( w = 10^p \in Q \) yields \( 10^p110^p1 = (10^p1)^2 \) by inserting a 1 before and after the second copy of \( 10^p \). Furthermore, if we cancel the first letter and insert a 1 before the last 0 in the duplication \( 110110 \) of \( 110 \in Q \), we get \( 1101110 = (110)^2 \not\in Q \), again.
Therefore we have a complete picture for the case that the edit distance is at most 2.

4 Concatenation of an Almost Mirror Image

In this section, again, we consider words of the form $ww'$. However, instead of an almost copy $w'$ of $w$ we choose $w'$ in such a way that the Hamming/edit distance of $w'$ and the mirror image $w^R$ is small.

We start with the remark that, in general, for a primitive word $w$, $ww^R$ is not a primitive word. For example, if we concatenate 100110 and its mirror image, we obtain $100110011001 = (1001)^3 \notin Q$. Moreover, if we delete one letter in $w^R$, the obtained operation is not primitivity preserving as can be seen from the following counterexample. Let $w = 01001$. Since $w^R = 10010$, $ww^R = 0100110010$. If we delete the first letter of $w^R$, then we obtain $010010010 = (010)^3 \notin Q$.

We define formally three operations which are analogous to some with a small Hamming distance $d(w,w')$ considered in the preceding section.

Definition 15 For any natural numbers $n, i, j$ with $1 \leq i \leq n$ and $2 \leq j \leq n$, all letters $x, y \in V$ with $x \neq y$, and a word $w = x_1x_2...x_n$, $x_i \in V$, of length $n$, we define the following operations

$$M_{n,i,x,y} : V^n \rightarrow V^{2n}, \quad \text{and} \quad M'_{n,j,x,y} : V^n \rightarrow V^{2n-1}$$

by

$$M_{n,i,x,y}(x_1x_2...x_n) = \begin{cases} x_1x_2...x_nx_{n-i+1}y_{x_i}x_{i-1}x_{i-2}...x_1 & x_i = x \\ \text{undefined} & \text{otherwise} \end{cases},$$

$$M'_{n,j,x,y}(x_1x_2...x_n) = \begin{cases} x_1x_2...x_nx_{n-j+1}y_{x_j}x_{j-1}x_{j-2}...x_2 & x_j = x \\ \text{undefined} & \text{otherwise} \end{cases}.$$

For all odd natural numbers $n$, all mappings $h : V \rightarrow V$ with $h(a) \neq a$ for all $a \in V$, and all words $w = x_1x_2...x_n$, $x_i \in V$, of length $n$, we define the operation $O'_{n,h} : V^n \rightarrow V^{2n}$ by

$$O'_{n,h}(x_1x_2...x_n) = x_1x_2...x_nh(x_n)x_{n-1}...x_{i+1}h(x_i)x_{i-1}x_{i-2}...x_2h(x_1)$$

where $i = \frac{n+1}{2}$.

Theorem 16 If $w \in Q$ such that $M_{n,i,x,y}(w)$ is defined, then $M_{n,i,x,y}(w) \in Q$ also holds.
Proof. Let $w = x_1 x_2 \ldots x_n$. Then

$$w' = M_{n,i,x,y}(w) = x_1 x_2 \ldots x_{i-1} x x_{i+1} x x_n x_{n-1} \ldots x_{i+1} y x_{i-1} x_{i-2} \ldots x_1.$$  

Let $u_1 = x_1 \ldots x_{i-1}$ and $u_2 = x_{i+1} \ldots x_n$. Then

$$w = u_1 x u_2 \quad \text{and} \quad w' = u_1 x u_2 u_R^2 y u_1^R.$$  

Let us assume that $w' \notin Q$. Then $w' = v^p$ for some $p \geq 2$ and some word $v \in V^+$.  

If $p$ is even, then

$$v^2 = u_1 x u_2 = u_2^R y u_1^R.$$  

We now count the number of occurrences of $x$ and get

$$\#_x(u_1 x u_2) = \#_x(u_1) + 1 + \#_x(u_2)$$  

and

$$\#_x(u_2^R y u_1^R) = \#_x(u_2^R) + \#_x(u_1^R) = \#_x(u_2) + \#_x(u_1).$$  

Thus

$$\#_x(u_1 x u_2) \neq \#_x(u_2^R y u_1^R)$$

which contradicts (3).

If $p$ is odd, say $p = 2m + 1$ for some $m \geq 1$, then $w' = v^m v_1 v_2 v^m$ where $v = v_1 v_2$ and $|v_1| = |v_2|$. If $i > |v|$, then by the construction of $w'$ we get $w' = v z v^R$ with $z = v^{m-1} v_1 v_2 v^{m-1}$ and by our assumption ($w' = v^{2m+1}$) we have $w' = v z v$. Therefore $v = v^R$. Now let $i \leq |v|$. Then $v_1$ and $v_2$ and $v$ satisfy the following conditions:

- $v_2 = v^R_1$ (by construction),
- $v^2_R = ((v_1^R)^R)^R = v_1$,
- $v^R = (v_1 v_2)^R = v_2^R v_1^R = v_1 v_2 = v$.

Hence in both cases we have $v = v^R$. This implies

$$(w')^R = (v^p)^R = (v^R)^p = v^p = w'.$$

Thus $x = y$ in contrast to our supposition. \qed

\textbf{Theorem 17} If $w \in Q$ such that $M_{n,i,x,y}^I(w)$ is defined, then $M_{n,i,x,y}^I(w) \in Q$ also holds.
Proof. Let \( w = x_1x_2 \ldots x_n \). Then

\[ M'_{n,j,x,y}(w) = x_1x_2 \ldots x_nx_{n-1} \ldots x_{j+1}y_{x_{j-1}x_{j-2}} \ldots x_2. \]

Obviously, \( |M'_{n,j,x,y}(w)| = 2n - 1 \), i.e., the length of \( M'_{n,j,x,y}(w) \) is odd.

If \( M'_{n,j,x,y}(w) \) is not a primitive word, then \( M'_{n,j,x,y}(w) = v^p \) for some primitive word \( v \) of odd length and some odd number \( p \) with \( p \geq 3 \), say \( p = 2m + 1 \) with \( m \geq 1 \). As in the preceding proofs we get \( v = v_1x_nv_2 \) with

\[ M'_{n,j,x,y}(w) = v^mv_1x_nv_2v^m = (v_1x_nv_2)^mv_1x_nv_2(v_1x_nv_2)^m \]

and \( |v_1| = |v_2| \). Let \( |v_1| = q \), i.e., \( |v| = 2q + 1 \).

Let \( 2 \leq j \leq 2q + 1 \). Then considering the \((m+1)\)-st factor \( v \) of \( M'_{n,j,x,y}(w) \), we obtain \( v = v_1x_nv_2 = x_1x_2 \ldots x_qx_n | x_nx_{n-1} \ldots x_2 \). Let \( z = x_2x_3 \ldots x_qx_n \). Then \( v = x_1zz^R \). On the other hand, for \( 2 \leq j \leq 2q + 1 \), by definition of \( M'_{n,j,x,y}(w) \), \( M'_{n,j,x,y}(w) \) does not end with \( (zz^R)^R = zz^R \). Thus we have a contradiction to the fact that \( M'_{n,j,x,y}(w) \) ends with \( v \) and therefore with \( zz^R \).

Let \( j = 2q + 2 \). Then the \((2q+2)\)-nd letter of \( w \) is \( x \). Moreover, the \((2q+2)\)-nd letter of \( w \) is the first letter of the second factor \( v \) of \( M'_{n,j,x,y}(w) \) which is \( x_1 \). Hence \( x = x_1 \). On the other hand, by the definition of \( M'_{n,j,x,y}(w) \), counting from the end, \( y \) is the \((2q+1)\)-st letter of \( M'_{n,j,x,y}(w) \), which means that \( y \) is the first letter of the last factor \( v \) of \( M_{n,j,x,y}(w) \). Thus \( y = x_1 \). Hence we get \( x = y \) in contradiction to the definition of \( M'_{n,j,x,y} \).

Let \( 2q+3 \leq j \leq n \). Then we can derive a contradiction by analogous argument (if \( m(2q+1) < j \leq n \), then we get \( v = v_1x_nv_2 = x_1zz^R \) by considering the first factor \( v_1 \) and the last factor \( v_2 \) in \( M'_{n,j,x,y}(w) \)). \( \square \)

Finally in this section, we give a result which is the counterpart of Theorem 9. We omit the proof which can be given in analogy to the proof of Theorem 9.

**Theorem 18** For any odd natural number \( n \geq 5 \), any primitive word \( q \) of length \( n \), and any mapping \( h : V \to V \) with \( h(a) \neq a \) for all \( a \in V \), \( O_{n,h}(q) \) is a primitive word. \( \square \)

### 5 Further Operations with an Almost Duplication of Length

First in this section, we discuss the situation where \( w' \) in \( ww' \) is obtained from \( w \) or \( w^R \) by large changes.

If we change all letters in the second part, primitivity is not preserved in general. For instance, if we take the primitive word \( w = 100110 \), then by changing all letters of \( w \) we obtain \( 100110011001 = (1001)^3 \notin Q \); and starting with the primitive word \( w = 10010110 \) and changing all letters of \( w^R \) we get \( 1001011010010110 = w^2 \notin Q \).
Thus let $D$ have an odd length.

Since $w$ does not hold for the mirror image, i.e., if one concatenates $w$ with its mirror image and changes all letters of the mirror image besides one letter, in general, one does not obtain a primitive word. For example, if $w = 1110011100 = (11100)^2 \not\in Q$. However, if we restrict to

**Theorem 19** Let $w$ and $w'$ be two words of length $n$ such that $n - d(w, w')$ is a power of 2, then $ww'$ is a primitive word.

**Proof.** The proof can be given in a way analogous to the proof of Theorem 7. □

The following definition and result are analogies to $D_{n,n}$ and Theorem 11.

**Definition 20** For any natural numbers $n$, any natural number $i$ with $1 \leq i \leq n$, and any homomorphism $h : V^* \rightarrow V^*$ with $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in V$, we define the operation $D_{n,h} : V^n \rightarrow V^{2n - 1}$ by

$$D_{n,h}(x_1x_2 \ldots x_n) = x_1x_2 \ldots x_nh(x_1x_2 \ldots x_{n-1}).$$

**Theorem 21** For any natural numbers $n$, any natural number $i$ with $1 \leq i \leq n$, any homomorphism $h : V^* \rightarrow V^*$ with $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in V$, and any $w \in Q$, $D_{n,h}(w) \in Q$ also holds.

**Proof.** Let $w = x_1x_2 \ldots x_n$ with $x_j \in V$ for $1 \leq j \leq n$. Then

$$D_{n,h}(x_1x_2 \ldots x_n) = x_1x_2 \ldots x_nh(x_1 \ldots x_{n-1})$$

has an odd length.

Let us suppose that $D_{n,h}(w) \not\in Q$, that is, there exist a $p \geq 2$ and $v \in Q$ such that $D_{n,h}(w) = v^p$.

Thus $p$ is odd, say $p = 2m + 1$ for some $m \geq 1$. As above there are words $v$, $v_1$ and $v_2$ such that $v = v_1x_nv_2$ and

$$x_1x_2 \ldots x_n|v(h(x_1 \ldots x_{n-1}) = (v_1x_nv_2)^mv_1x_nv_2(v_1x_nv_2)^m.$$ 

Since $|(v_1x_nv_2)^mv_1| = |v_2(v_1x_nv_2)^m|, |v_1| = |v_2|.$

Furthermore $v_2 = h(v_1)$ by definition of $D_{n,h}$. Therefore we get

$$x_1x_2 \ldots x_n|v(h(x_1 \ldots x_{n-1}) = (v_1x_wh(v_1))^mv_1x_n|v(h(v_1)(v_1x_nh(v_1)))^m.$$ 

Thus $(h(v_1)h(x_n)v_1)^m(h(v_1) = h(v_1)(v_1x_nh(v_1))^m$, that is,

$$(h(v_1)h(x_n)v_1)^m(h(v_1) = (h(v_1)v_1x_nh(v_1))^m(h(v_1)).$$

Hence $h(x_n)v_1 = v_1x_n$. Therefore, by Lemma 6, $h(x_n) = x_n$ in contrast to the supposition concerning $h$. □

By Theorem 19, from a word $w \in Q$ we obtain a primitive word $ww'$ where $w'$ is constructed from $w$ by changing all letters except one letter. This result does not hold for the mirror image, i.e., if one concatenates $w$ with its mirror image and changes all letters of the mirror image besides one letter, in general, one does not obtain a primitive word. For example, if $w = 11100 \in Q$ and $i = 3$, then we obtain $1110011100 = (11100)^2 \not\in Q$. However, if we restrict to
special positions, then the corresponding statement is true, as shown by the following two theorems.

**Definition 22** For any natural numbers $n$ and $i$ with $1 \leq i \leq n$ and any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ for all $a \in V$, we define the operations

$$M_{n,1,h}, \ M_{n,n,h} : V^n \to V^{2n}$$

by

$$M_{n,1,h}(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n h(x_{n-1} x_{n-2} \ldots x_1),$$

$$M_{n,n,h}(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n h(x_n x_{n-1} \ldots x_2) x_1.$$  

**Theorem 23** For any $n \geq 2$, any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ for all $a \in V$ and any $w \in Q$, $M_{n,1,h}(w) \in Q$ also holds.

**Proof.** Let $w = x_1 x_2 \ldots x_n$, where $x_i \in V$. Then

$$M_{n,1,h}(w) = x_1 x_2 \ldots x_n \ x_{n-1} x_n h(x_{n-1} x_{n-2} \ldots x_1)$$

has an even length.

Let us suppose that $M_{n,1,h}(w) \notin Q$, that is, there exists a $p \in \mathbb{N}$ and $v \in Q$ such that $x_1 x_2 \ldots x_{n-1} x_n x_n h(x_{n-1} x_{n-2} \ldots x_1) = v^p$.

If $p$ is even and $p > 2$, then $v^{\frac{p}{2}} = w$ and $\frac{p}{2} \geq 2$, which contradicts $w \in Q$. If $p = 2$, then $x_1 x_2 \ldots x_{n-1} x_n x_n h(x_{n-1} x_{n-2} \ldots x_1) = v^2$, that is, $v = x_1 x_2 \ldots x_n x_n = h(x_{n-1} x_{n-2} \ldots x_1)$. Then $x_n = x_1$ and $x_n = h(x_1)$, which is a contradiction.

If $p$ is odd, then $p = 2m + 1$ for some $m \geq 1$ and $v = x_1 v' x_n v''$ with $v', v'' \in V^*$, which can be shown as in the proof of Theorem 12. Since

$$x_1 \ldots x_{n-1} x_n x_n h(x_{n-1} x_{n-2} \ldots x_1) = v^m x_1 v' x_n v'' v^m, \ |v'| = |v''|.$$  

We distinguish the cases $v' \neq \lambda \neq v''$ and $v' = \lambda = v''$.

Supposing $v' \neq \lambda \neq v''$ and $v' = y_1 \ldots y_r$ and $v'' = z_1 \ldots z_r$. Then

$$x_1 \ldots x_{n-1} x_n |x_n h(x_{n-1} x_{n-2} \ldots x_1) = (x_1 y_1 \ldots y_r x_n z_1 \ldots z_r)^m x_1 y_1 \ldots y_r |x_n z_1 \ldots z_r x_1 y_1 \ldots y_r x_n z_1 \ldots z_r)^m$$

and $y_r = x_n$. Since $h(x_1 y_1 y_2 \ldots y_r) = z_r z_{r-1} \ldots z_1 x_n$ by construction, $h(y_r) = x_n$, which contradicts $y_r = x_n$.

Supposing $v' = \lambda = v''$, we get

$$x_1 \ldots x_{n-1} x_n |x_n h(x_{n-1} x_{n-2} \ldots x_1) = (x_1 x_n)^m x_1 |x_n (x_1 x_n)^m,$$

which implies $x_n = x_1$ and $x_n = h(x_1)$, so it is a contradiction.

Therefore $Q_{n,1,h}(w) \in Q$. □
Theorem 24 For any \( n \geq 2 \), any homomorphism \( h : V^* \rightarrow V^* \) with \( h(a) \neq a \) for all \( a \in V \) and any \( w \in Q \), \( M_{n,n,h}(w) \in Q \) also holds.

Proof. Let \( w = x_1x_2\ldots x_n \). Let us assume that \( M_{n,n,h}(w) \notin Q \). Then there is a word \( v \in V^+ \) and a natural number \( p \geq 2 \) such that \( M_{n,n,h}(w) = v^p \).

If \( p = 2 \), then \( v = x_1x_2\ldots x_n = h(x_nx_{n-1}\ldots x_2)x_1 \). Hence \( x_1 = h(x_n) \) and \( x_n = x_1 \), which is a contradiction. If \( p > 2 \) and even, then \( w = v^{\frac{2}{2}} \in Q \) in contrast to our supposition.

If \( p \) is odd, i.e., \( p = 2m + 1 \) for some \( m \geq 1 \), then there are words \( v_1 \) and \( v_2 \) with \( v = v_1v_2 \), \( |v_1| = |v_2| \) and

\[
x_1x_2\ldots x_n|h(x_nx_{n-1}\ldots x_2)x_1 = v^mv_1|v_2v^m.
\]

Let \( k = |v_1| \). Then

\[
v_1 = x_1x_2\ldots x_k \quad \text{and} \quad v_2 = h(x_kx_{k-1}\ldots x_2)x_1
\]

by definition of \( M_{n,n,h} \). Thus \( x_{2k+1} = x_1 \) and \( h(x_{2k+1}) = x_1 \) in contrast to the required property of \( h \) that \( h(a) \neq a \) for all \( a \in V \).

We now define an operation where we duplicate the word, but the copy is shifted some positions to the left. Hence, on one hand, no change is done in the copy, but on the other hand, the position of the letters are changed essentially. An analogous operation is performed where we shift an almost completely changed version of the word.

Definition 25 For any natural numbers \( n \) and \( i \) with \( 1 \leq i \leq n-1 \), we define the operation \( S_{n,i} : V^n \rightarrow V^{2n} \) by

\[
S_{n,i}(x_1x_2\ldots x_n) = x_1x_2\ldots xi_1x_2\ldots xn_{x_i+1}x_{i+2}\ldots xn.
\]

Theorem 26 For any natural numbers \( n \geq 2 \) and \( i \) with \( 1 \leq i \leq n-1 \) and any word \( q \in Q \) of length \( n \), \( S_{n,i}(q) \in Q \) also holds.

Proof. Let \( q = w w' \in Q \) with \( w = x_1x_2\ldots xi_{-1} \) and \( w' = xi_{i+1}\ldots xn \), where \( x_j \in V \) for \( 1 \leq j \leq n \). Then \( S_{n,i}(q) = www' \).

Assume \( www'w' \notin Q \), that is, there exist a \( p \in \mathbb{N} \), \( p > 2 \) and \( v \in Q \) such as \( www'w' = v^p \), that is, \( w^2(w')^2 = v^p \). It is known, by Lemma 3, \( w = u^k \) and \( v = u^m \). Since \( www' \in Q \) and \( w^2w' = u^{k+l} \), we have a contradiction.

Therefore \( www'w' \in Q \).
Finally in the following theorem we present some operations which, together with the above operations, allow the generation of all primitive words of length \( \leq 11 \) (as can be shown by computer calculations) and of a considerable amount of primitive words of length up to twenty.

**Theorem 27** Let \( w \in Q \) be a primitive word of length \( n \geq 2 \) and \( x \in V \) and \( y \in V \) two different letters of \( V \).

i) Then \( wx^n \) and \( wx^{n-1} \) and \( wxy^{n-2} \) are in \( Q \), too.

ii) If \( n \) is even, then \( w(xy)^{(n-2)/2}x \) and \( w(xy)^{(n-2)/2}y \) are primitive words, too.

**Proof.** We omit the easy proofs for i).

ii) We only prove the statement for \( w(xy)^{(n-2)/2}x \); the other proof can be given analogously.

Let us assume that \( w(xy)^{(n-2)/2}x \notin Q \). Then there is a word \( v \in V^+ \) such that \( w(xy)^{(n-2)/2}x = v^p \) for some \( p \geq 2 \). Since \( w(xy)^{(n-2)/2}x \) has odd length, \( p \) and the length of \( v \) are odd numbers. Let \( p = 2m + 1 \) for some \( m \geq 1 \). Thus there are \( v_1, v_2 \in V^+ \) such that

\[
v = v_1v_2, \quad |v_1| = |v_2| + 1 \quad \text{and} \quad w|(xy)^{(n-2)/2}x = v^m v_1 |v_2 v^n.\]

By \( w(xy)^{(n-2)/2}x = v^{2m+1} \), \( v = (xy)^k x \) for some \( k \geq 1 \), and then \( v_1 = (xy)^r \), \( v_2 = (xy)^{r-1} x \) and

\[
w|(xy)^{(n-2)/2}x = ((xy)^k x)^m(xy)^r |(xy)^{r-1} x ((xy)^k x)^m.\]

Since the \( (n + 2(r - 1) + 2) \)-nd letters in both representations differ, we have a contradiction. \( \Box \)

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**References**


