Some Operations Preserving Primitivity of Words

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Abstract

We investigate some operations where essentially, from a given word w, the word ww' is constructed where w' is a modified copy of w or a modified mirror image of w. We study whether ww' is a primitive word provided that w is primitive. For instance, we determine all cases with an edit distance of w and w' at most 2 such that the primitivity of w implies the primitivity of ww'. The operations are chosen in such a way that in the case of a two-letter alphabet, all primitive words of length ≤ 11 can be obtained from single letters.

1 Introduction

A word w over an alphabet V is said to be a primitive word if and only if there is no word $u \in \Sigma^+$ with $w = u^n$ for some natural number $n > 1$. The set of all primitive words over V is denoted by Q_V. There are a lot of papers on relations of Q_V to other language families as the families of the Chomsky hierarchy (e.g. in [4] and [17], it has been shown that Q_V is neither a deterministic nor an unambiguous context-free language, in [8] relations to regular languages are given), Marcus contextual grammars (see [6]), to (poly-)slender languages (see [5]) and some languages and language families related to codes

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(see e.g. [19]). Moreover, there are papers on combinatorial properties of primitive words and of the sets Q_V; we refer to [2], [1], [9].

However, there is only a small number of results concerning the closure of Q_V under operations. There are some papers where it was investigated whether the application of homomorphisms to primitive words leads to primitive words in all cases or leads to primitive words with a finite number of exceptions or to non-primitive words in all cases; we refer to [13], [14], [15], [10]. In [18] homomorphisms are studied which preserve the property to be a Lyndon word or to be border-free (a word w is a Lyndon word if and only if any non-empty proper suffix of w is greater than w with respect to the lexicographic order; it is border-free if there is no non-empty word which is a proper prefix as well as a proper suffix of w); it is shown that such homomorphisms preserve primitivity, too. Substitutions form another operation which was investigated with respect to preservation of primitivity. There were substitutions of very short subwords in the focus, especially point mutations (deletions, insertions and substitutions of one letter) were studied. We refer to [16] for details. A further study in this direction concerns insertions (see [11]).

Obviously, there is a large variety of operations from which one can expect that Q_V is closed under them (since the portion of primitive words is very high). In this paper we consider some operations where essentially, from a given word w, the word ww' is constructed where w' is a modified copy of w or a modified mirror image of w. The modifications are of such a form that the edit distance of w and w' is very small or very large (i.e., it is very near to the length of w).

We have two reasons for this investigation. The first one is of combinatorial nature. Obviously, ww is not primitive for all w. We are interested in conditions for changes of the second copy w to w' such that ww' is primitive for all w. Especially, how many changes or deletions or insertions of letters are necessary and how many such operations are possible. For example, we shall determine all possible transformation where the edit distance of w and w' is at most two and primitivity is preserved.
The second reason comes from the theory of dynamical systems. In the paper [7] a dynamical system based on regular languages has been proposed. The regular languages are essentially described by primitive words. Since in dynamical systems one needs mutations in order to develop the system, one is interested in devices which describe primitive words and allow mutations. Here the use of operations which preserve primitivity is of interest. Then a primitive word can be given as a sequence of operations; and a mutation is the replacement of one operation by another one or a deletion or insertion of an operation in the sequence. This ensures primitivity of the word obtained from the mutated sequence of operations. Obviously, it is not necessary to generate all primitive words, however, the set of generated primitive words should contain a good approximation of any primitive word where the quality of approximations is determined by the dynamic system (especially its fitness

function). We have chosen the operations under which Q_V is closed in such a way that, if the underlying alphabet V consists of two letters, then by the operations we can generate all primitive words of length ≤ 11 (as can be shown by computer calculations) and a sufficient large amount of primitive words of the length up to twenty.
Thus this paper can also be considered as a continuation of the investigations of devices generating only primitive words (see e.g. [3]).

The paper is organized as follows. In Section 2, we present and recall some notations and some results on primitive words which are used in the sequel. In Section 3, we introduce some operations where we first construct ww and perform then some small modifications of the second copy yielding ww'. We prove that all operations where the edit distance of w and w' is 1 preserve primitivity. An analogous result is shown for the edit distance 2 if at least one change of a letter is used. In Section 4, we consider analogous operations as in Section 2, but start from ww^R and modify w^R. In Section 5 we consider ww' where w' is obtained from w or w^R by a drastic change, i.e., the Hamming distance of w' and w or w^R is almost the length of w. Moreover, we give some further operations where the length is almost doubled and primitivity is preserved.

2 Some Notation and Facts

By $\#(A)$ we denote the cardinality of a set A.
For a given alphabet V, we denote by V^* and V^+ the set of all and all non-empty words over V, respectively. The empty word is designated by λ. Given a word $w \in V^*$ and $x \in V$, we denote its length by $|w|$ and the number of occurrences of x in w by $\#_x(w)$. For a word $w = x_1x_2 \dots x_n \in V^+$ with $x_i \in V$ for $1 \leq i \leq n$, we define the mirror image w^R by $w^R = x_nx_{n-1} \dots x_1$. Given two words $w = x_1x_2 \dots x_n \in V^+$ and $w' = y_1y_2 \dots y_n \in V^+$ with $x_i, y_i \in V$ for $1 \leq i \leq n$, the Hamming distance $d(w, w')$ is defined by $d(w, w') = \#(\{i \mid x_i \neq y_i\})$ and the edit distance $ed(w, w')$ of w and w' is the minimal number of changes, deletions and insertions of letters in order to transform w into w'. Throughout the paper we assume that V has at least two elements.

A word $w \in V^+$ is said to be a primitive word if and only if there is no word $u \in V^+$ such that $w = u^n$ for some natural number $n > 1$. By Q_V we denote the set of all primitive words over V. If V is understood from the context we omit the index V and write simply Q.

We recall three facts (see [12], [19], [1]) which will be used in the sequel.

Lemma 1 *For any words $v, v' \in V^*$, $vv' \in Q$ if and only if $v'v \in Q$.*

Lemma 2 *For two non-empty words u and v, $uv = vu$ if and only if there is* *a word z such that $u = z^n$ and $v = z^m$ for some natural numbers n and m.* $\square$

Lemma 3 *In a free monoid V^*, the equation $a^m b^n = c^p$, where $a, b, c \in V^*$ and $m, n, p \geq 2$, has only trivial solutions, where a, b and c are powers of some word in V^*.* $\square$

Lemma 4 (Fine-Wilf Theorem) *Let $u, v \in V^+$ and $n, m \geq 2$. If u^n and v^m have a common prefix of length at least $|u| + |v| - gcd(|u|, |v|)$, then u and v are powers of the same primitive word.* $\square$

The following statement holds trivially.

Lemma 5 *If $w \in Q$, then also $w^R \in Q$.* $\square$

Lemmas 1 and 5 can be interpreted as follows: If we apply a cyclic shift or the mirror image to a primitive word, then we obtain a primitive word, again. Thus cyclic shifts and reversal are operations which preserve primitivity.

Lemma 6 *For any $x \in V$, $y \in V$ and $z \in V^*$, if $xz = zy$, then $x = y$.*

Proof. If $z = \lambda$, then $x = y$ immediately. If $z = a_1 a_2 \dots a_n$ with $a_i \in V$ for $1 \leq i \leq n$, then $x = a_1, a_1 = a_2, a_2 = a_3, \dots a_{n-1} = a_n, a_n = y$ and consequently $x = y$. $\square$

In the sequel we shall use the following notation. If $w = w_1 w_2 \dots w_r = z_1 z_2 \dots z_s$ for some words $w_1, \dots w_r, z_1, \dots, z_s \in V^*$ such that $|w_1 w_2 \dots w_i| = |z_1 z_2 \dots z_j|$ for some i and j, we write

$$w_1 w_2 \dots w_i | w_{i+1} w_{i+2} \dots w_r = z_1 z_2 \dots z_j | z_{j+1} z_{j+2} \dots z_s,$$

i.e., by the symbol $|$ we mark a certain position in the word. (Some authors write $(w, w') = (z, z')$ instead of $w|w' = z|z'$.) Mostly, $|$ will mark the middle of a word of even length, or it will be put after the m-th letter if the word has odd length $2m - 1$.

3 Operations with an Almost Duplication

Obviously, the word ww obtained from w by a duplication leads from any word w to a non-primitive word. In order to obtain primitive words from a primitive word w one has to perform some changes in the second occurrence of w, i.e., one has to consider words of the form ww' where w' differs only slightly from w. In most cases the edit distance of w and w' will be at most 2, and thus ww' can be considered as an almost duplication of w.
We start with the case where we only change some letters to obtain w' from w.

Theorem 7 *i) Let w be a primitive word of some length n and w' an arbitrary word of length n such that the Hamming distance $d(w, w')$ is a power of 2, then ww' is primitive, too.*
ii) If d is not a power of 2, then there are a primitive word w and a word w' with $d(w, w') = d$ such that ww' is not a primitive word.

Proof. i) Obviously, $|ww'|$ is even. Let us suppose $ww' \notin Q$, that is, there exists $p \in \mathbb{N}$ and $v \in V^+$ of length at least 2 such that $ww' = v^p$.
If p is even, then $w = w' = v^{\frac{p}{2}}$ since $|w| = |w'|$. Thus $d(w, w') = 0$ which contradicts the assumption on the Hamming distance of w and w'.
If p is odd, i.e., $p = 2m + 1$ for some $m \geq 1$, then $|v|$ is even (since otherwise $|v|p = |ww'|$ would be odd). Thus there are words v' and v'' of length $\frac{|v|}{2}$ such that $v = v'v''$. Then we get $w = v^m v' = (v'v'')^m v'$ and $w' = v''v^m = v''(v'v'')^m$. Then $d(w, w') = (2m + 1)d(v', v'')$. Since $2m + 1$ is an odd number, $d(w, w')$ is not a power of 2 in contrast to our supposition.
ii) Let d be not a power of 2. Then there is an odd number $q > 1$ and a number p such that $d = qp$. Let $q = 2m + 1$ for some $m \geq 1$. We now set

$$v' = 10^p, \quad v'' = 11^p, \quad w = (v'v'')^m v', \quad \text{and} \quad w' = (v''v')^m v''.$$

Obviously, w is primitive, $d(w, w') = (2m + 1)d(v', v'') = (2m + 1)p = qp = d$ and $ww' = (v'v'')^{2m+1} \notin Q$. $\square$

By part ii) of the preceding theorem, if w is a primitive word and $d(w, w')$ is not a power of 2, in general, ww' is not a primitive word. However, if we require that the changes occur in special positions it is possible to obtain preservation of primitivity. As an example we give the following operation.

Definition 8 *For any odd natural numbers $n \geq 3$, any alphabet V, and any mapping $h : V \to V$ with $h(a) \neq a$ for all $a \in V$, we define the operation $O_{n,h} : V^n \to V^{2n}$ by*

$$O_{n,h}(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n h(x_1) x_2 \dots x_{i-1} h(x_i) x_{i+1} \dots x_{n-1} h(x_n)$$

where $i = \frac{n+1}{2}$.

Theorem 9 *For any odd natural number $n \neq 5$, any primitive word q of length n, and any mapping $h : V \to V$ with $h(a) \neq a$ for all $a \in V$, $O_{n,h}(q)$ is a primitive word.*

Proof. Let $w = x_1 x_2 \dots x_n$ with $x_j \in V$ for $1 \leq j \leq n$ and $i = \frac{n+1}{2}$. Then $O_{n,h}(x_1 x_2 \dots x_n)$ has an even length.
Let us suppose that $O_{n,h}(w) \notin Q$, that is, there exist a $p \geq 2$ and $v \in Q$ such that $O_{n,h} = v^p$.
If p is even then

$$v^{\frac{p}{2}} = x_1 x_2 \dots x_{n-1} x_n = h(x_1) x_2 x_3 \dots x_{i-1} h(x_i) x_{i+1} x_{i+2} \dots x_{n-1} h(x_n).$$

Thus $x_i = h(x_i)$, which is a contradiction.
Thus p is odd, say $p = 2m + 1$ for some $m \geq 1$. As above there are words v, v_1 and v_2 such that $v = v_1 v_2$ and $|v_1| = |v_2|$ and

$$x_1 \dots x_{n-1} x_n | h(x_1) x_2 \dots x_{i-1} h(x_i) x_{i+1} \dots x_{n-1} h(x_n) = (v_1 v_2)^m v_1 | v_2 (v_1 v_2)^m.$$

Since v_1 starts with x_1 (first occurrence) and ends with x_n (last occurrence in the first part), $v_1 = x_1 v_1' x_n$ and analogously, $v_2 = h(x_1) v_2' h(x_n)$. Therefore we have that $O_{n,h}(w)$ has the form

$$(x_1 v_1' x_n h(x_1) v_2' h(x_n))^m x_1 v_1' x_n | h(x_1) v_2' h(x_n) (x_1 v_1' x_n h(x_1) v_2' h(x_n))^m.$$

Since the letters x_i and x_n do not occur in the first occurrence of v, by the definition of $O_{n,h}$, the last letter of the first occurrence of v_1 (in the first part of the word) and last letter of the first occurrence of v_2 in the second part coincide, i.e., $x_n = h(x_n)$ which is a contradiction. $\square$

The supposition $n \geq 5$ in Theorem 9 is necessary since the statement does not hold for $n = 3$ as can be seen from the following example. Let $q = aba \in Q$. Then $O_{3,h}(q) = ababab = (ab)^3 \notin Q$.

We now discuss some operations where the edit distance of w to w' is at most 2 and at least one deletion or one insertion is performed to obtain w'; more precisely, we consider
(a) the deletion of an arbitrary letter,
(b) the deletion of an arbitrary letter and the change of an arbitrary remaining letter,
(c) the insertion of an arbitrary letter,
(d) the insertion of an arbitrary letter and the change of an arbitrary letter of w.
We now give the formal definition of these operations.

Definition 10 *For any natural numbers n, i, j, i' with $1 \leq i \leq n$, $0 \leq i' \leq n$, $1 \leq j \leq n$ and $i \neq j$, letters $x, y, z \in V$ with $x \neq y$, and a word $w = x_1 x_2 \dots x_n$, $x_i \in V$, of length n, we define the following operations*

$$D_{n,i},\ D_{n,i,j,x,y} : V^n \to V^{2n-1} \text{ and } I_{n,i',z},\ I_{n,i',z,j,x,y} : V^n \to V^{2n+1}$$

by

$$D_{n,i}(x_1x_2\dots x_n) = x_1x_2\dots x_nx_1x_2\dots x_{i-1}x_{i+1}x_{i+2}\dots x_n,$$

$$D_{n,i,j,x,y}(x_1\dots x_n) = \begin{cases} x_1\dots x_nx_1\dots x_{i-1}x_{i+1}\dots x_{j-1}yx_{j+1}\dots x_n & x_j = x, i < j \\ x_1\dots x_nx_1\dots x_{j-1}yx_{j+1}\dots x_{i-1}x_{i+1}\dots x_n & x_j = x, i > j \\ \textit{undefined} & \textit{otherwise} \end{cases},$$

$$I_{n,i',z}(x_1x_2\dots x_n) = x_1x_2\dots x_nx_1x_2\dots x_{i'}zx_{i'+1}x_{i'+2}\dots x_n,$$

$$I_{n,i',z,j,x,y}(x_1\dots x_n) = \begin{cases} x_1\dots x_nx_1\dots x_{i'}zx_{i'+1}\dots x_{j-1}yx_{j+1}\dots x_n & x_j = x, i' < j \\ x_1\dots x_nx_1\dots x_{j-1}yx_{j+1}\dots x_{i'}zx_{i'+1}\dots x_n & x_j = x, i' > j \\ \textit{undefined} & \textit{otherwise} \end{cases}.$$

Theorem 11 *If $n \geq 2$, $1 \leq i \leq n$, and q is a primitive word of length n, then $D_{n,i}(q) \in Q$ also holds.*

Proof. Let $q = uav$ for some $u, v \in V^*$ and $a \in V$. If $|q| = 1$, i.e., $q = a$, then $D_{n,i}(q) = a \in Q$.
If $|q| \geq 2$, then $D_{|q|,|u|+1}(q) = uavuv$. Let us suppose that $uavuv \notin Q$. Then $(vu)^2a \notin Q$ by Lemma 1. Let $(vu)^2a = z^m$ for some $z \in V^+$ and some $m \geq 2$. Thus $(vu)^2$ is a common prefix of $(vu)^2$ and z^m. Since

$$|uv| + |z| = |uv| + \frac{2|uv|+1}{m} \leq |vu| + \frac{2|uv|+1}{2} < 2|vu| + 1,$$

we have

$$|vu| + |z| - gcd(|vu|, |z|) < (2|vu| + 1) - 1 = 2|vu| = |(vu)^2|.$$

By Lemma 4, we obtain $(vu)^2 = w^k$ and $(vu)^2a = z^m = w^l$ for some $w \in V^+$ and some numbers k and l. Obviously, $w = a$. Hence u and v are powers of a and thus q is a power of a. This contradicts the primitivity of q. □

Theorem 12 *If $w \in V^+$ and $D_{n,i,j,x,y}(w)$ is defined, then $D_{n,i,j,x,y}(w) \in Q$ holds.*

Proof. We first discuss $D_{n,n,j,x,y}$. Let $w = x_1x_2\dots x_n$. Then

$$D_{n,n,j,x,y}(w) = x_1x_2\dots x_{j-1}xx_{j+1}x_{j+2}\dots x_nx_1x_2\dots x_{j-1}yx_{j+1}x_{j+2}\dots x_{n-1}.$$

Let us assume that $D_{n,n,j,x,y}(w) \notin Q$. Then there is a word $v \in V^+$ such that $D_{n,n,j,x,y}(w) = v^p$ for some $p \geq 2$. Since $D_{n,n,j,x,y}(w)$ has odd length, p and the length of v are odd numbers. Let $p = 2m + 1$ for some $m \geq 1$. Thus there are words $v_1 \in V^+$ and $v_2 \in V^+$ such that $v = x_1v_1v_2$, $k - 1 = |v_1| = |v_2|$ and

$$x_1x_2\dots x_{j-1}xx_{j+1}x_{j+2}\dots x_n|x_1x_2\dots x_{j-1}yx_{j+1}x_{j+2}\dots x_{n-1} = v^mx_1v_1|v_2v^m.$$

Then $|v| = 2k - 1$. We set $s = 2k - 1$. We distinguish some cases.
Case 1. Let $1 \le j \le k - 1$. Then by definition of $D_{n,n,j,x,y}$,

$$x_1 v_1 = x_1 x_2 \dots x_{j-1} x x_{j+1} \dots x_{k-1} x_k = z_1 x z_2 x_k$$

and

$$v_2 = x_1 x_2 \dots x_{j-1} y x_{j+1} \dots x_{k-1} = z_1 y z_2.$$

Thus, we get,

$$v = z_1 x z_2 x_k z_1 y z_2.$$

If $m \ge 2$, the first part of the word is

$$z_1 x z_2 x_k z_1 y z_2 z_1 x z_2 x_k z_1 y z_2 v^{m-2} z_1 x z_2 x_k \tag{1}$$

and that of the second part is

$$z_1 y z_2 z_1 x z_2 x_k z_1 y z_2 z_1 x z_2 x_k z_1 y z_2 v^{m-2} \tag{2}$$

and these two words differ in the $(|z_1 x z_2 x_k z_1 y z_2 z_1| + 1)$-st letter, which contradicts the definition of $D_{n,n,j,x,y}$. If $m = 1$, the first and second part are

$$z_1 x z_2 x_k z_1 y z_2 z_1 x z_2 x_k \text{ and } z_1 y z_2 z_1 x z_2 x_k z_1 y z_2,$$

respectively, and we get a contradiction as above.
Case 2. Let $j = k$. Then the k-th letter in the second part is y. On the other hand, it is x_1 since there starts the word v. Thus $x_1 = y$. This gives

$$x_1 v_1 = x_1 x_2 \dots x_{k-1} x_k = yzx, \;\; v_2 = x_1 x_2 \dots x_{k-1} = yz \text{ and } v = yzxyz$$

with $z = x_2 x_3 \dots x_{k-1}$. Then the first and second part are

$$yzxyzyzxyzv^{m-2}yzx \text{ and } yzyzxyzyzxyzv^{m-2},$$

respectively. We obtain $zx = yz$ by looking on the words starting in the position $|z| + 3$. Thus by Lemma 6, $x = y$ in contrast to the definition of $D_{n,n,j,x,y}$.
Case 3. Let $k+1 \le j \le 2k-1$. Then $v = x_1 v_1 v_2' x v_2''$. Moreover, $|v_2'| = j-k-1$. Furthermore, y stands in the j-th position of $v_2' x v_2'' x_1 v_1$, i.e., $x_1 v_1 = x_1 v_1' y v_1''$ with $|v_1'| = j-k-1$. Therefore $v = x_1 v_1' y v_1'' v_2' x v_1''$ and $|v_1'| = |v_2'|$ and $|v_1''| = |v_2''|$. Then we get for the second part

$$x_1 v_1' y v_1'' v_2' y v_2'' x_1 v_1' y v_1'' v_2' x v_2'' x_{2s-1} x_{2s} \dots x_n$$

by the definition of $D_{n,n,j,x,y}$ and from the form

$$v_2' x v_2'' x_1 v_1' y v_1'' v_2' x v_2'' v^{m-1}$$

given by our assumption. Considering the words which start in the position $(|x_1v_1'yv_1''|+1)$ and in the position $(|x_1v_1'yv_1''v_2'y|+1)$, respectively, we see that $v_1' = v_2' = z$ and $v_1'' = v_2'' = z'$. Looking on the subwords starting in the first position and in the position $|v_1'| + 2$, we get $x_1z = zx$ and $yz' = xx_1$. By Lemma 6, $x_1 = x$ and $y = x_1$, which contradicts $x \neq y$.

Case 4. Let $j = hs + q$ for some $h \geq 1$ and $1 \leq q \leq k - 1$. Then $x_j = x$ is the q-th letter of v. Thus $v = v_1'xv_1''v_2$ with $|v_1'| = q - 1$.
We now compute the position of y in v. Since the second part starts with v_2 of length $k-1$ and $hs+q = k-1+(h-1)s+s+q-(k-1) = k_1+(h-1)s+k+q$, y is the $(k+q)$-th letter of v. Therefore $v = v_1'xv_1''v_2'yv_2''$ with $|v_1'| = |v_2'|$. Moreover, $|v_1''| = |v_2''|+1$. Now we get easily the same situation as in Case 1; thus we get (1) and (2) and a difference in the $(|z_1|+1)$-st position.

Case 5. Let $j = hs + k$ for some $h \geq 1$. Then x is the k-th letter of v. We compute the position of y in v. Since the second part starts with v_2 of length $k-1$ and $hs+k = k-1+hs+k-(k-1)$, y is the first letter of v. Therefore we get $v = yzxyz$ as in Case 2, which leads to a contradiction.

Case 6. Let $j = hs + q$ for some $h \geq 1$ and $k + 1 \leq q \leq 2k - 1$. Then $x_j = x$ is the q-th letter of v. Thus $v = x_1v_1v_2'xv_2''$ with $|x_1v_1v_2'| = q - 1 \geq k$. Furthermore, $|v_2''| = 2k - 1 - q$.
We now compute the position of y in v. Since the second part starts with v_2 of length $k - 1$ and $hs + q = k - 1 + hs + q - (k - 1)$, y is the $(q - k + 1)$-st letter of v. Therefore

$$v = x_1v_1'yv_1''v_2'xv_2'' \text{ with } |x_1v_1'| = q - k.$$

Therefore $|v_1''| = k-(q-k+1) = 2k-1-q$. Hence $|v_1''| = |v_2''|$ and consequently $|v_1'| = |v_2'|$. Therefore we have exactly the situation of Case 3, which leads to contradiction.

Let us now consider $i = 1$, i.e., the operation $D_{n,1,j,x,y}$. By the first part of this proof

$$D_{n,n,n-j+1,x,y}(w^R) = x_nx_{n-1}\dots x_1x_nx_{n-1}\dots x_{j+1}yx_{j-1}x_{j-2}\dots x_2 \in Q,$$

by Lemma 5,

$$x_2x_3\dots x_{j-1}yx_{j+1}x_{j+2}\dots x_nx_1x_2\dots x_n \in Q,$$

and by Lemma 1

$$x_1x_2\dots x_nx_2x_3\dots x_{j-1}yx_{j+1}x_{j+2}\dots x_n = D_{n,1,j,x,y}(w) \in Q.$$

We now consider the case $j < i$. We set

$$\overline{w} = x_{i+1}x_{i+2}\dots x_nx_1x_2\dots x_i.$$

Moreover, let $x_j = x$. By the first part of this proof we get

$$D_{n,n,n-i+j,x,y}(\overline{w}) = x_{i+1} \dots x_n x_1 \dots x_i x_{i+1} \dots x_n x_1 \dots x_{j-1} y x_{j+1} \dots x_{i-1} \in Q.$$

Hence, by Lemma 1

$$x_1 \dots x_i x_{i+1} \dots x_n x_1 \dots x_{j-1} y x_{j+1} \dots x_{i-1} x_{i+1} \dots x_n = D_{n,i,j,x,y}(w) \in Q.$$

If $i < j$ we can prove that $D_{n,i,j,x,y}(w) \in Q$ analogously to the case $j < i$ using $D_{n,1,j,x,y}$ instead of $D_{n,n,j,x,y}$. $\square$

Theorem 13 *If q is a primitive word of length n, $0 \leq i \leq n$ and $z \in V$, then $I_{n,i,z}(q) \in Q$.*

Proof. Let q be a primitive word of length n and $a \in V$. Let u be the prefix of q of length i and $q = uv$. Then $I_{n,i,a}(w) = uvuav$. If $uvuav \notin Q$, we can derive a contradiction as in the proof of Theorem 11. $\square$

Theorem 14 *If $q \in Q$ and $I_{n,i,z,j,x,y}(q)$ is defined, then $I_{n,i,z,j,x,y}(q) \in Q$.*

Proof. Let $w = x_1 x_2 \dots x_{j-1} x x_{j+1} x_{j+2} \dots x_n$. Then

$$I_{n,n,a,j,x,y} = x_1 x_2 \dots x_n x_1 x_2 \dots x_{j-1} y x_{j+1} x_{j+2} \dots x_n a.$$

If we assume that $I_{n,n,a,j,x,y}$ is not in Q, then

$$x_1 \dots x_{j-1} y x_{j+1} \dots x_n a x_1 \dots x_n = D_{n+1,n+1,j,y,x}(x_1 \dots x_{j-1} y x_{j+1} \dots x_n a) \notin Q,$$

which is a contradiction to Theorem 12. The general case can be obtained using Lemmas 1 and 5. $\square$

Let ww' be given with $ed(w, w') = 1$. Then w' is obtained by a change (i.e., $d(w, w') = 1 = 2^0$), either by a deletion or by an insertion. By the Theorems 7, 11 and 13, ww' is in Q provided that $w \in Q$. If $ed(w, w') = 2$ we have again $ww' \in Q$ if two changes, or a deletion and a change, or a change and an insertion are performed (by Theorems 7, 12 and 14). In the remaining cases, in general, primitivity is not preserved. Performing two deletions we can get a non-primitive word, as can be seen from $w = 110^p1$ which results in 110^p1110^p1 and gives $110^p110^p = (110^p)^2 \notin Q$ if we delete the first and last letters of the second copy (note that the statement holds for any length $n \geq 4$ since it holds for any $p \geq 1$). The same holds for two insertions; e.g. the duplication 10^p10^p of $w = 10^p \in Q$ yields $10^p110^p1 = (10^p1)^2$ by inserting a 1 before and after the second copy of 10^p. Furthermore, if we cancel the first letter and insert a 1 before the last 0 in the duplication 110110 of $110 \in Q$, we get $110110 = (110)^2 \notin Q$, again.

Therefore we have a complete picture for the case that the edit distance is at most 2.

4 Concatenation of an Almost Mirror Image

In this section, again, we consider words of the form ww'. However, instead of an almost copy w' of w we choose w' in such a way that the Hamming/edit distance of w' and the mirror image w^R is small.
We start with the remark that, in general, for a primitive word w, ww^R is not a primitive word. For example, if we concatenate 100110 and its mirror image, we obtain $100110011001 = (1001)^3 \notin Q$. Moreover, if we delete one letter in w^R, the obtained operation is not primitivity preserving as can be seen from the following counterexample. Let $w = 01001$. Since $w^R = 10010$, $ww^R = 0100110010$. If we delete the first letter of w^R, then we obtain $010010010 = (010)^3 \notin Q$.
We define formally three operations which are analogous to some with a small Hamming distance $d(w, w')$ considered in the preceding section.

Definition 15 *For any natural numbers n, i, j with $1 \leq i \leq n$ and $2 \leq j \leq n$, all letters $x, y \in V$ with $x \neq y$, and a word $w = x_1x_2 \ldots x_n$, $x_i \in V$, of length n, we define the following operations*

$$M_{n,i,x,y} : V^n \to V^{2n}, \text{ and } M'_{n,j,x,y} : V^n \to V^{2n-1}$$

by

$$M_{n,i,x,y}(x_1x_2 \ldots x_n) = \begin{cases} x_1x_2 \ldots x_nx_nx_{n-1} \ldots x_{i+1}yx_{i-1}x_{i-2} \ldots x_1 & x_i = x \\ \mathit{undefined} & \mathit{otherwise} \end{cases},$$

$$M'_{n,j,x,y}(x_1x_2 \ldots x_n) = \begin{cases} x_1x_2 \ldots x_nx_nx_{n-1} \ldots x_{j+1}yx_{j-1}x_{j-2} \ldots x_2 & x_j = x \\ \mathit{undefined} & \mathit{otherwise} \end{cases}.$$

For all odd natural numbers n, all mappings $h : V \to V$ with $h(a) \neq a$ for all $a \in V$, and all words $w = x_1x_2 \ldots x_n$, $x_i \in V$, of length n, we define the operation $O'_{n,h} : V^n \to V^{2n}$ by

$$O'_{n,h}(x_1x_2 \ldots x_n) = x_1x_2 \ldots x_nh(x_n)x_{n-1} \ldots x_{i+1}h(x_i)x_{i-1}x_{i-2} \ldots x_2h(x_1)$$

where $i = \frac{n+1}{2}$.

Theorem 16 *If $w \in Q$ such that $M_{n,i,x,y}(w)$ is defined, then $M_{n,i,x,y}(w) \in Q$ also holds.*

Proof. Let $w = x_1 x_2 \dots x_n$. Then

$$w' = M_{n,i,x,y}(w) = x_1 x_2 \dots x_{i-1} x x_{i+1} x_{i+2} x_n x_n x_{n-1} \dots x_{i+1} y x_{i-1} x_{i-2} \dots x_1.$$

Let $u_1 = x_1 \dots x_{i-1}$ and $u_2 = x_{i+1} \dots x_n$. Then

$$w = u_1 x u_2 \text{ and } w' = u_1 x u_2 u_2^R y u_1^R.$$

Let us assume that $w' \notin Q$. Then $w' = v^p$ for some $p \geq 2$ and some word $v \in V^+$.
If p is even, then

$$v^{\frac{p}{2}} = u_1 x u_2 = u_2^R y u_1^R. \tag{3}$$

We now count the number of occurrences of x and get

$$\#_x(u_1 x u_2) = \#_x(u_1) + 1 + \#_x(u_2)$$

and

$$\#_x(u_2^R y u_1^R) = \#_x(u_2^R) + \#_x(u_1^R) = \#_x(u_2) + \#_x(u_1).$$

Thus

$$\#_x(u_1 x u_2) \neq \#_x(u_2^R y u_1^R)$$

which contradicts (3).
If p is odd, say $p = 2m+1$ for some $m \geq 1$, then $w' = v^m v_1 v_2 v^m$ where $v = v_1 v_2$ and $|v_1| = |v_2|$. If $i > |v|$, then by the construction of w' we get $w' = vzv^R$ with $z = v^{m-1} v_1 v_2 v^{m-1}$ and by our assumption ($w' = v^{2m+1}$) we have $w' = vzv$. Therefore $v = v^R$. Now let $i \leq |v|$. Then v_1 and v_2 and v satisfy the following conditions:

- $v_2 = v_1^R$ (by construction),
- $v_2^R = ((v_1)^R)^R = v_1$,
- $v^R = (v_1 v_2)^R = v_2^R v_1^R = v_1 v_2 = v$.

Hence in both cases we have $v = v^R$. This implies

$$(w')^R = (v^p)^R = (v^R)^p = v^p = w'.$$

Thus $x = y$ in contrast to our supposition. $\square$

Theorem 17 *If $w \in Q$ such that $M'_{n,j,x,y}(w)$ is defined, then $M'_{n,i,x,y}(w) \in Q$ also holds.*

Proof. Let $w = x_1x_2\ldots x_n$. Then

$$M'_{n,j,x,y}(w) = x_1x_2\ldots x_nx_nx_{n-1}\ldots x_{j+1}yx_{j-1}x_{j-2}\ldots x_2.$$

Obviously, $|M'_{n,j,x,y}(w)| = 2n-1$, i.e., the length of $M'_{n,j,x,y}(w)$ is odd.
If $M'_{n,j,x,y}(w)$ is not a primitive word, then $M'_{n,j,x,y}(w) = v^p$ for some primitive word v of odd length and some odd number p with $p \geq 3$, say $p = 2m+1$ with $m \geq 1$. As in the preceding proofs we get $v = v_1x_nv_2$ with

$$M'_{n,j,x,y}(w) = v^m v_1x_n|v_2v^m = (v_1x_nv_2)^m v_1x_n|v_2(v_1x_nv_2)^m$$

and $|v_1| = |v_2|$. Let $|v_1| = q$, i.e., $|v| = 2q+1$.
Let $2 \leq j \leq 2q+1$. Then considering the $(m+1)$-st factor v of $M'_{n,j,x,y}(w)$, we obtain $v = v_1x_n|v_2 = x_1x_2\ldots x_qx_n|x_nx_q\ldots x_2$. Let $z = x_2x_3\ldots x_qx_n$. Then $v = x_1zz^R$. On the other hand, for $2 \leq j \leq 2q+1$, by definition of $M'_{n,j,x,y}(w)$, $M'_{n,j,x,y}(w)$ does not end with $(zz^R)^R = zz^R$. Thus we have a contradiction to the fact that $M'_{n,j,x,y}(w)$ ends with v and therefore with zz^R.
Let $j = 2q+2$. Then the $(2q+2)$-nd letter of w is x. Moreover, the $(2q+2)$-nd letter of w is the first letter of the second factor v of $M'_{n,j,x,y}(w)$ which is x_1. Hence $x = x_1$. On the other hand, by the definition of $M'_{n,j,x,y}(w)$, counting from the end, y is the $(2q+1)$-st letter of $M'_{n,j,x,y}(w)$, which means that y is the first letter of the last factor v of $M_{n,j,x,y}(w)$. Thus $y = x_1$. Hence we get $x = y$ in contradiction to the definition of $M'_{n,j,x,y}$.
Let $2q+3 \leq j \leq n$. Then we can derive a contradiction by analogous argument (if $m(2q+1) < j \leq n$, then we get $v = v_1x_nv_2 = x_1zz^R$ by considering the first factor v_1 and the last factor v_2 in $M'_{n,j,x,y}(w)$). □

Finally in this section, we give a result which is the counterpart of Theorem 9. We omit the proof which can be given in analogy to the proof of Theorem 9.

Theorem 18 *For any odd natural number $n \geq 5$, any primitive word q of length n, and any mapping $h : V \to V$ with $h(a) \neq a$ for all $a \in V$, $O'_{n,h}(q)$ is a primitive word.* □

5 Further Operations with an Almost Duplication of Length

First in this section, we discuss the situation where w' in ww' is obtained from w or w^R by large changes.
If we change all letters in the second part, primitivity is not preserved in general. For instance, if we take the primitive word $w = 100110$, then by changing all letters of w we obtain $100110011001 = (1001)^3 \notin Q$; and starting with the primitive word $w = 10010110$ and changing all letters of w^R we get $1001011010010110 = w^2 \notin Q$.

Theorem 19 *Let w and w' be two words of length n such that $n - d(w, w')$ is a power of 2, then ww' is a primitive word.*

Proof. The proof can be given in a way analogous to the proof of Theorem 7. □

The following definition and result are analogies to $D_{n,n}$ and Theorem 11.

Definition 20 *For any natural numbers n, any natural number i with $1 \leq i \leq n$, and any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in V$, we define the operation $D_{n,h} : V^n \to V^{2n-1}$ by*

$$D_{n,h}(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n h(x_1 x_2 \dots \dots x_{n-1}).$$

Theorem 21 *For any natural numbers n, any natural number i with $1 \leq i \leq n$, any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in V$, and any $w \in Q$, $D_{n,h}(w) \in Q$ also holds.*

Proof. Let $w = x_1 x_2 \dots x_n$ with $x_j \in V$ for $1 \leq j \leq n$. Then

$$D_{n,h}(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n h(x_1 \dots x_{n-1})$$

has an odd length.
Let us suppose that $D_{n,h}(w) \notin Q$, that is, there exist a $p \geq 2$ and $v \in Q$ such that $D_{n,h}(w) = v^p$.
Thus p is odd, say $p = 2m + 1$ for some $m \geq 1$. As above there are words v, v_1 and v_2 such that $v = v_1 x_n v_2$ and

$$x_1 x_2 \dots x_n | h(x_1 \dots x_{n-1}) = (v_1 x_n v_2)^m v_1 x_n | v_2 (v_1 x_n v_2)^m.$$

Since $|(v_1 x_n v_2)^m v_1| = |v_2 (v_1 x_n v_2)^m|$, $|v_1| = |v_2|$.
Furthermore $v_2 = h(v_1)$ by definition of $D_{n,h}$. Therefore we get

$$x_1 x_2 \dots x_n | h(x_1 \dots x_{n-1}) = (v_1 x_n h(v_1))^m v_1 x_n | h(v_1)(v_1 x_n h(v_1))^m.$$

Thus $(h(v_1) h(x_n) v_1)^m h(v_1) = h(v_1)(v_1 x_n h(v_1))^m$, that is,

$$(h(v_1) h(x_n) v_1)^m h(v_1) = (h(v_1) v_1 x_n)^m h(v_1).$$

Hence $h(x_n) v_1 = v_1 x_n$. Therefore, by Lemma 6, $h(x_n) = x_n$ in contrast to the supposition concerning h. □

By Theorem 19, from a word $w \in Q$ we obtain a primitive word ww' where w' is constructed from w by changing all letters except one letter. This result does not hold for the mirror image, i.e., if one concatenates w with its mirror image and changes all letters of the mirror image besides one letter, in general, one does not obtain a primitive word. For example, if $w = 11100 \in Q$ and $i = 3$, then we obtain $1110011100 = (11100)^2 \notin Q$. However, if we restrict to special positions, then the corresponding statement is true, as shown by the following two theorems.

Definition 22 *For any natural numbers n and i with $1 \le i \le n$ and any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ for all $a \in V$, we define the operations*

$$M_{n,1,h},\ M_{n,n,h} : V^n \to V^{2n}$$

by

$$\begin{aligned} M_{n,1,h}(x_1x_2\dots x_n) &= x_1x_2\dots x_nx_nh(x_{n-1}x_{n-2}\dots x_1),\\ M_{n,n,h}(x_1x_2\dots x_n) &= x_1x_2\dots x_nh(x_nx_{n-1}\dots x_2)x_1. \end{aligned}$$

Theorem 23 *For any $n \ge 2$, any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ for all $a \in V$ and any $w \in Q$, $M_{n,1,h}(w) \in Q$ also holds.*

Proof. Let $w = x_1x_2\dots x_n$, where $x_i \in V$. Then

$$M_{n,1,h}(w) = x_1x_2\dots x_{n-1}x_nx_nh(x_{n-1}x_{n-2}\dots x_1)$$

has an even length.
Let us suppose that $M_{n,1,h}(w) \notin Q$, that is, there exists a $p \in \mathbb{N}$ and $v \in Q$ such that $x_1x_2\dots x_{n-1}x_nx_nh(x_{n-1}x_{n-2}\dots x_1) = v^p$.
If p is even and $p > 2$, then $v^{\frac{p}{2}} = w$ and $\frac{p}{2} \ge 2$, which contradicts $w \in Q$. If $p = 2$, then $x_1x_2\dots x_{n-1}x_nx_nh(x_{n-1}x_{n-2}\dots x_1) = v^2$, that is, $v = x_1x_2\dots x_{n-1}x_n = x_nh(x_{n-1}x_{n-2}\dots x_1)$. Then $x_n = x_1$ and $x_n = h(x_1)$, which is a contradiction.
If p is odd, then $p = 2m+1$ for some $m \ge 1$ and $v = x_1v'x_nv''$ with $v', v'' \in V^*$, which can be shown as in the proof of Theorem 12. Since

$$x_1\dots x_{n-1}x_n|x_nh(x_{n-1}x_{n-2}\dots x_1) = v^mx_1v'|x_nv''v^m,\ |v'| = |v''|.$$

We distinguish the cases $v' \neq \lambda \neq v''$ and $v' = \lambda = v''$.
Supposing $v' \neq \lambda \neq v''$ and $v' = y_1\dots y_r$ and $v'' = z_1\dots z_r$. Then

$$\begin{aligned} &x_1\dots x_{n-1}x_n|x_nh(x_{n-1}x_{n-2}\dots x_1)\\ &\quad = (x_1y_1\dots y_rx_nz_1\dots z_r)^mx_1y_1\dots y_r|x_nz_1\dots z_r(x_1y_1\dots y_rx_nz_1\dots z_r)^m \end{aligned}$$

and $y_r = x_n$. Since $h(x_1y_1y_2\dots y_r) = z_rz_{r-1}\dots z_1x_n$ by construction, $h(y_r) = x_n$, which contradicts $y_r = x_n$
Supposing $v' = \lambda = v''$, we get

$$x_1\dots x_{n-1}x_n|x_nh(x_{n-1}x_{n-2}\dots x_1) = (x_1x_n)^mx_1|x_n(x_1x_n)^m,$$

which implies $x_n = x_1$ and $x_n = h(x_1)$, so it is a contradiction.
Therefore $Q_{n,1,h}(w) \in Q$. $\square$

Theorem 24 *For any $n \geq 2$, any homomorphism $h : V^* \to V^*$ with $h(a) \neq a$ for all $a \in V$ and any $w \in Q$, $M_{n,n,h}(w) \in Q$ also holds.*

Proof. Let $w = x_1x_2 \dots x_n$. Let us assume that $M_{n,n,h}(w) \notin Q$. Then there is a word $v \in V^+$and a natural number $p \geq 2$ such that $M_{n,n,h}(w) = v^p$.
If $p = 2$, then $v = x_1x_2 \dots x_n = h(x_nx_{n-1} \dots x_2)x_1$. Hence $x_1 = h(x_n)$ and $x_n = x_1$, which is a contradiction. If $p > 2$ and even, then $w = v^{\frac{p}{2}} \in Q$ in contrast to our supposition.
If p is odd, i.e., $p = 2m + 1$ for some $m \geq 1$, then there are words v_1 and v_2 with $v = v_1v_2$, $|v_1| = |v_2|$ and

$$x_1x_2 \dots x_n|h(x_nx_{n-1} \dots x_2)x_1 = v^m v_1|v_2 v^m.$$

Let $k = |v_1|$. Then

$$v_1 = x_1x_2 \dots x_k \quad \text{and} \quad v_2 = h(x_kx_{k-1} \dots x_2)x_1$$

by definition of $M_{n,n,h}$. Thus $x_{2k+1} = x_1$ and $h(x_{2k+1}) = x_1$ in contrast to the required property of h that $h(a) \neq a$ for all $a \in V$. □

We now define an operation where we duplicate the word, but the copy is shifted some positions to the left. Hence, on one hand, no change is done in the copy, but on the other hand, the position of the letters are changed essentially. An analogous operation is performed where we shift an almost completely changed version of the word.

Definition 25 *For any natural numbers n and i with $1 \leq i \leq n-1$, we define the operation $S_{n,i} : V^n \to V^{2n}$ by*

$$S_{n,i}(x_1x_2 \dots x_n) = x_1x_2 \dots x_i x_1x_2 \dots x_n x_{i+1}x_{i+2} \dots x_n.$$

Theorem 26 *For any natural numbers $n \geq 2$ and i with $1 \leq i \leq n-1$ and any word $q \in Q$ of length n, $S_{n,i}(q) \in Q$ also holds.*

Proof. Let $q = ww' \in Q$ with $w = x_1x_2 \dots x_{i-1}$ and $w' = x_ix_{i+1} \dots x_n$, where $x_j \in V$ for $1 \leq j \leq n$. Then $S_{n,i}(q) = www'w'$.
Assume $www'w' \notin Q$, that is, there exist a $p \in \mathbb{N}$, $p > 2$ and $v \in Q$ such as $www'w' = v^p$, that is, $w^2(w')^2 = v^p$. It is known, by Lemma 3, $w = u^k, w' = u^l, v = u^m$. Since $ww' \in Q$ and $ww' = u^{k+l}$, we have a contradiction.
Therefore $www'w' \in Q$. □

We mention that an analogous statement does not hold, if one uses the mirror image instead of a copy. The following example shows that then primitivity is not preserved. Let $w = 01$ and $i = 1$; using the mirror image and shifting it by one position to the left we get $0101 \notin Q$.

Finally in the following theorem we present some operations which, together with the above operations, allow the generation of all primitive words of length ≤ 11 (as can be shown by computer calculations) and of a considerable amount of primitive words of length up to twenty.

Theorem 27 *Let $w \in Q$ be a primitive word of length $n \geq 2$ and $x \in V$ and $y \in V$ two different letters of V.*
i) Then wx^n and wx^{n-1} and wxy^{n-2} are in Q, too.
ii) If n is even, then $w(xy)^{(n-2)/2}x$ and $w(xy)^{(n-2)/2}y$ are primitive words, too.

Proof. We omit the easy proofs for i).
ii) We only prove the statement for $w(xy)^{(n-2)/2}x$; the other proof can be given analogously.
Let us assume that $w(xy)^{(n-2)/2}x \notin Q$. Then there is a word $v \in V^+$ such that $w(xy)^{(n-2)/2}x = v^p$ for some $p \geq 2$. Since $w(xy)^{(n-2)/2}x$ has odd length, p and the length of v are odd numbers. Let $p = 2m+1$ for some $m \geq 1$. Thus there are $v_1, v_2 \in V^+$ such that

$$v = v_1v_2,\ |v_1| = |v_2| + 1 \text{ and } w|(xy)^{(n-2)/2}x = v^m v_1|v_2 v^m.$$

By $w(xy)^{(n-2)/2}x = v^{2m+1}$, $v = (xy)^k x$ for some $k \geq 1$, and then $v_1 = (xy)^r$, $v_2 = (xy)^{r-1}x$ and

$$w|(xy)^{(n-2)/2}x = ((xy)^k x)^m (xy)^r|(xy)^{r-1}x((xy)^k x)^m.$$

Since the $(n+2(r-1)+2)$-nd letters in both representations differ, we have a contradiction. $\square$

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