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Numerical Integration of Lattice Systems with a Lyapunov Function

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Abstract

In this contribution we implement and assess numerical methods for gradient systems, i.e. dynamical systems that possess a Lyapunov function, and consequently are stable. In particular, we claim that discrete gradient methods are well suited to so-called lattice systems, i.e. systems of ordinary differential equations that can reach high dimensionality. For these systems, reproducing the stable qualitative behaviour is more important than achieving an overly accurate quantitative approximation. The presented results show that discrete gradient methods outperform conventional Runge-Kutta methods, since these latter algorithms destroy the stability of the original system.

Key words: Gradient Systems, Geometric Numerical Integration, Lyapunov Function, Discrete Gradient, Lattice Systems
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1 Introduction

The main aim of this paper is to analyse the performance of numerical methods specifically designed to preserve the Lyapunov function of a stable dynamical system. In particular, we deal with higher-dimensional systems of Ordinary Differential Equations (ODEs), which can be regarded as abstract models of lattice systems arising in a wide variety of fields of physics, biology and engineering, e.g. the Ising model in statistical mechanics, and the Toda lattice in crystallography. Although some of these systems present a cyclic behaviour, it is often the case that the system converges towards a stable state while, at the same time, the energy diminishes.

From a mathematical point of view, the energy-diminishing feature amounts to proving that the system possesses a Lyapunov function, which in turn is a proof for the asymptotic stability of fixed points (see e.g. [3] for references and background). In other words, the distinguishing characteristic of a Lyapunov function is that it decreases along the system trajectories and the main goal of this work is to reproduce this behaviour under discretization by a numerical method. ODEs with a Lyapunov function V receive the name of *gradient systems* because they can be rewritten as

$$\frac{d\mathbf{y}}{dt} = L(\mathbf{y})\,\nabla V(\mathbf{y}) \tag{1}$$

where $\nabla V(y)$ is the gradient of V and L is a negative definite matrix.

Within the framework of Geometric Numeric Integration [1], many methods for ODEs have been designed to preserve qualitative features. However, most of these proposals are related to symplectic methods, i.e. methods for Hamiltonian systems. In contrast, numerical methods for gradient systems are scarce. In this regard, because of its conceptual simplicity, discrete gradient methods [4] are appealing, but there is still limited experience as to their relative strengths and limitations. This paper aims at exploring a discrete gradient method, by undertaking its implementation and application to a particular class of lattice systems.

2 Discrete gradient methods for Hopfield neural networks

Discrete gradient methods consist in approximating Equation (1) by the iteration $\frac{\mathbf{y_{n+1}}-\mathbf{y_n}}{h} = \widetilde{L}(\mathbf{y_n}, \mathbf{y_{n+1}}, h)\ \overline{\nabla} V(\mathbf{y_n}, \mathbf{y_{n+1}})$, where $\widetilde{L}$ and $\overline{\nabla} V$ are discretizations of L and ∇V, respectively, which must fulfil the consistency conditions $\widetilde{L}(\mathbf{y}, \mathbf{y}, 0) = L(\mathbf{y})$ and $\overline{\nabla} V(\mathbf{y}, \mathbf{y}) = \nabla V(\mathbf{y})$. Besides, the discrete gradient is defined by the requirement $\overline{\nabla} V(\mathbf{y_n}, \mathbf{y_{n+1}}) \cdot (\mathbf{y_{n+1}} - \mathbf{y_n}) = V(\mathbf{y_{n+1}}) - V(\mathbf{y_n})$. It is worth emphasizing that there is considerable freedom in the choice of $\widetilde{L}$ and $\overline{\nabla} V$, so one of the contributions of this work is to determine which values of these design parameters result in more favourable solutions. We have implemented the *coordinate increment discrete gradient*, whereas the matrix $\widetilde{L}$ is assumed to be identical to L. It turns out that this choice allows for obtaining an explicit method for particular cases of ODEs.

Continuous Hopfield neural networks [2] were proposed as a model of biological neurons, given by the system of ODEs: $\frac{du_i}{dt} = \sum_{j=1}^{n} w_{ij}\, y_j - b_i\,,\ \ y_i = \tanh(u_i)$. These networks may be defined for a high dimension n, and they possess a multilinear Lyapunov function $V(\mathbf{y}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\, y_i\, y_j + \sum_{i=1}^{n} b_i\, y_i$. After eliminating the internal variables u_i, this model can be cast into the gradient form of Equation (1). The numerical experiments that have been performed show that the discrete gradient method preserves the energy-diminishing feature of Hopfield networks, whereas the Euler rule and other conventional Runge-Kutta methods fail to reproduce the expected stable behaviour. Also, the implemented discrete gradient method is computationally favourable, since it can be written in explicit form.

References

[1] E. Hairer, *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*, Springer, 2010.

[2] J.J. Hopfield, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proceedings of the National Academy of Science **81**(10) (1984) 3088–3092.

[3] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, 2002.

[4] R.I. McLachlan, G.R.W. Quispel, N. Robidoux, *Geometric integration using discrete gradients*, Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **357**(1754) (1999) 1021–1045.