The two weight problem for the Bergman projection and Sarason Conjecture
(Joint work with A. Aleman and S. Pott)

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The Bergman space on the disc

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The Bergman space $L^p_a(\mathbb{D}) := \{ f \in L^p(\mathbb{D}) : f \text{ is analytic} \}$. 
Let $P_B$ be the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. The operator is known as the Bergman Projection and it can be written as

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**Theorem**

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**Theorem**

$P_B : L^p(\mathbb{D}) \rightarrow L^p_a(\mathbb{D})$ bounded for $1 < p < \infty$. 
We define $P_B^+$, the so called maximal Bergman Projection, as

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$$P_B^+ f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{|1 - z\bar{\xi}|^2} dA(\xi).$$

\textbf{Theorem} \quad P_B^+: L^p(\mathbb{D}) \to L^p(\mathbb{D}) \text{ bounded for } 1 < p < \infty,
The main question

Question

Given \( w \) and \( v \) two function weights (positive, locally integrable functions), find necessary and sufficient conditions for the boundedness of the Bergman projection \( P \) in the corresponding weighted spaces, i.e.,

\[
P : L^2(v, \mathbb{D}) \mapsto L^2(w, \mathbb{D}).
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An equivalent formulation:

\[
P(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D}),
\]

where \( \sigma = v^{-1} \).
Definition

Let $f \in L^\infty(\mathbb{D})$, we define the Toeplitz operator with symbol $f$ as

$$T_f(h) = P_B(fh), \quad h \in L^2_a(\mathbb{D})$$
Toeplitz Operators on $L^2_a(\mathbb{D})$

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**Remark:**

1. $T_g^* = T_{\bar{g}}$ and $T_f(h) = fh$ when $f$ is analytic.
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**Remark:**

1. $T^*_g = T_{\bar{g}}$ and $T_f(h) = fh$ when $f$ is analytic.
2. Toeplitz operators with analytic symbols are known to be bounded if and only if the symbol is bounded.
Let $f, g \in L^2_a(D)$

**Conjecture (Sarason)**

$$T_f T^*_g : L^2_a(D) \leftrightarrow L^2_a(D) \iff \sup_{z \in D} B(|f|^2)(z) B(|g|^2)(z) < \infty$$

where

$$B(h)(z) = (1 - |z|^2)^2 \int_D \frac{h(\xi)}{|1 - \bar{\xi}z|^4} dA(\xi)$$

is the so called **Berezin transform**.
Sarason Conjecture on the Bergman space

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The corresponding conjecture for $H^2$ is similar, one simply replaces the Berezin transforms by Poisson integrals.
The Berezin condition

The Berezin condition \( \sup_{z \in \mathbb{D}} B(|f|^2)(z) B(|g|^2)(z) < \infty \) is inspired by the Békollé-Bonami condition

\[
\sup_{I \subset \mathbb{T}} \frac{w(Q_I)}{|Q_I|} \left( \frac{w^{1-p'}(Q_I)}{|Q_I|} \right)^{p-1} < \infty,
\]

where \( Q_I \) is the Carleson box associated to \( I \),

\[
Q_I := \{ re^{i\theta} : 1 - |I| < r < 1, \ e^{i\theta} \in I \}.
\]
What was known

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What was known

1. Necessity of the Berezin condition was proved by Stroethoff and Zheng.

2. A "bumped Berezin condition" is sufficient (Stroethoff-Zheng and Michalska-Nowak-Sobolewski).

3. In the $H^2$ case, the Poisson condition is necessary (Treil). But unfortunately not sufficient (Nazarov).
Motivation: Sarason Conjecture on the Bergman space
The two weight problem for the Bergman projection
The Sarason case
Some open questions

An observation by Cruz-Uribe

\[ L^2_a(D) \xrightarrow{T_f T^*_g} L^2_a(D) \]

\[ \begin{array}{ccc}
L^2(1/|g|^2) & \xrightarrow{P_B} & L^2(|f|^2) \\
M_{\bar{g}} \downarrow & & \uparrow M_f \\
M_g & & 
\end{array} \]
An observation by Cruz-Uribe

Let $h \in L^2_a(\mathbb{D})$, $T_f T_g^*(h) = fP_B(\bar{g}h)$. 
Generalized Sarason Conjecture

Conjecture (Two weight Conjecture for the Bergman Projection)

Let $w, \sigma$ be two weights in $\mathbb{D}$ then

$$\sup_{z \in \mathbb{D}} B(w)(z)B(\sigma)(z) < \infty,$$  \hspace{1cm} (1)

if and only if

$$P_{B}(\sigma \cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w).$$ \hspace{1cm} (2)
Proposition

Let $w = (1 - |z|^2)^2$ and $\sigma$ be a weight, then the weights $w$ and $\sigma$ satisfy the Berezin condition if and only if

$$\int_{Q_I} \sigma dA \preceq \frac{1}{\log \frac{2}{|I|}} \quad \text{for all arcs } I \subset \mathbb{T}.$$
**Proposition**

Let \( w = (1 - |z|^2)^2 \) and \( \sigma \) be a weight, then the weights \( w \) and \( \sigma \) satisfies the Berezin condition if and only if

\[
\int_{Q_l} \sigma \, dA \lesssim \frac{1}{\log \frac{2}{|l|}} \quad \text{for all arcs } l \subset \mathbb{T}.
\]

On the other hand,

**Proposition**

If \( w(z) = (1 - |z|^2)^2, \ z \in \mathbb{D}, \) and \( \sigma \) a weight then

\[ P_B(\sigma \cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w) \] if and only if \( \sigma \, dA \) is a Carleson measure for the Dirichlet space.

Stegenga's counterexample fits in this framework.
Conjecture (Two weight Conjecture for the Bergman Projection)

Let \( w, \sigma \) be two weights in \( \mathbb{D} \), then the following are equivalent:

1. \( P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D}) \)

2. 
\[
\| P_B(\sigma 1_{Q_I}) \|_{L^2(w, \mathbb{D})} \leq C_0 \| 1_{Q_I} \|_{L^2(\sigma, \mathbb{D})},
\]

and
\[
\| P_B^*(w 1_{Q_I}) \|_{L^2(\sigma, \mathbb{D})} \leq C_0 \| 1_{Q_I} \|_{L^2(w, \mathbb{D})},
\]

for all intervals \( I \in \mathcal{T} \) and with constant \( C_0 \) uniform on \( I \).
A counterexample to Sarason Conjecture

Lemma

Let \( f \in L^2_a \), and let \( g \) be a Lipschitz analytic function in \( \mathbb{D} \) with \( |g(z)| \geq c(1 - |z|) \), for some constant \( c > 0 \) and all \( z \in \mathbb{D} \).

(i) If \( fg \in H^\infty \) and

\[
\int_{Q_l} |f|^2 \, dA \lesssim \frac{1}{\log \frac{2}{|l|}} \quad \text{for all arcs } l \subset \mathbb{T},
\]

then the Berezin condition holds.

(ii) If \( T_f T_g^* \) is bounded then \( |f|^2 \, dA \) is a Carleson measure for the Dirichlet space.
**Lemma**

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(ii) If \( T_f T_g^* \) is bounded then \( |f|^2 dA \) is a Carleson measure for the Dirichlet space.

The counterexample is based on Stegenga’s example, the key to finding such a \( g \) is in Dyn’kin’s work.
The two weight problem for $P_B$: the Sarason case

Theorem (Aleman, Pott, R.)

Let $f, g \in L^2_a(D)$ and consider the weights $\sigma = |g|^2$ and $w = |f|^2$. Then the following are equivalent

1. $P_B(\sigma \cdot) : L^2(\sigma, D) \mapsto L^2(w, D)$
2. 
\[
\| P_B^+(\sigma 1_{Q_I}) \|_{L^2(w, D)} \leq C_0 \| 1_{Q_I} \|_{L^2(\sigma, D)},
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for all intervals $I \in \mathbb{T}$ and with constant $C_0$ uniform on $I$. 

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Proof strategy

1. Prove a two weight estimate for $P_B^+$
   1. Find a dyadic model for $P_B^+$
   2. Use the two weight result for dyadic positive operators (Nazarov-Treil-Volber, Lacey-Sawyer-Uriarte-Tuero)

2. Prove the equivalence of boundedness of $P_B$ and $P_B^+$

Theorem (Aleman, Pott, R.)

Let $f, g \in L_a^2(\mathbb{D})$ and consider the weights $\sigma = |g|^2$ and $w = |f|^2$. Then the following are equivalent

1. $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$,
2. $P^+_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$
Open questions

1. Characterize the weights $w$ and $\sigma$ for which $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$. 

Are there other applications to the two weight problem for the Bergman projection other than Sarason conjecture?
Open questions

1. Characterize the weights $w$ and $\sigma$ for which $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$.

2. Are there other applications to the two weight problem for the Bergman projection other than Sarason conjecture?
MUCHAS GRACIAS!
THANK YOU!