The two weight problem for the Bergman projection and Sarason Conjecture

(Joint work with A. Aleman and S. Pott)

Maria Carmen Reguera

University of Birmingham, UK

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- The Bergman space $L^p_a(\mathbb{D}) := \{f \in L^p(\mathbb{D}) : f \text{ is analytic}\}$.

The Bergman projection

Let P_B be the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. The operator is known as the Bergman Projection and it can be written as

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The main question

Question

Given w and v two function weights (positive, locally integrable functions), find necessary and sufficient conditions for the boundedness of the Bergman projection P in the corresponding weighted spaces, i.e.,

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An equivalent formulation:

$$P(\sigma\cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D}),$$

where $\sigma = v^{-1}$.

Toeplitz Operators on $L^2_a(\mathbb{D})$

Definition

Let $f \in L^\infty(\mathbb{D})$, we define the Toeplitz operator with symbol f as

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Remark:

1. $T_g^* = T_{\bar{g}}$ and $T_f(h) = fh$ when f is analytic.
2. Toeplitz operators with analytic symbols are known to be bounded if and only if the symbol is bounded.

Sarason Conjecture on the Bergman space

Let $f, g \in L^2_a(\mathbb{D})$

Conjecture (Sarason)

$$T_f T_g^* : L^2_a(\mathbb{D}) \mapsto L^2_a(\mathbb{D}) \iff \sup_{z\in\mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty$$

where

$$B(h)(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{h(\xi)}{|1-\bar{\xi}z|^4} dA(\xi)$$

is the so called Berezin transform .

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The corresponding conjecture for H^2 is similar, one simply replaces the Berezin transforms by Poisson integrals.

The Berezin condition

The Berezin condition $\sup_{z\in\mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty$ is inspired by the Békollé-Bonami condition

$$\sup_{I\subset\mathbb{T}} \frac{w(Q_I)}{|Q_I|}\left(\frac{w^{1-p'}(Q_I)}{|Q_I|}\right)^{p-1} < \infty,$$

where Q_I is the Carleson box associated to I,
$Q_I := \{re^{i\theta} \ : \ 1-|I| < r < 1, \, e^{i\theta} \in I\}$.

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1. Necessity of the Berezin condition was proved by Stroethoff and Zheng.
2. A "bumped Berezin condition" is sufficient (Stroethoff-Zheng and Michalska-Nowak-Sobolewski).
3. In the H^2 case, the Poisson condition is necessary (Treil). But unfortunately not sufficient (Nazarov).

An observation by Cruz-Uribe

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\begin{array}{ccc}
L^2_a(\mathbb{D}) & \xrightarrow{\;T_f T_g^*\;} & L^2_a(\mathbb{D}) \\
{\scriptstyle M_{\bar{g}}}\big\downarrow & & \big\uparrow{\scriptstyle M_f} \\
L^2(1/|g|^2) & \xrightarrow{\;P_B\;} & L^2_a(|f|^2)
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Let $h \in L^2_a(\mathbb{D})$, $T_f T_g^*(h) = f P_B(\bar{g} h)$.

Generalized Sarason Conjecture

Conjecture (Two weight Conjecture for the Bergman Projection)

Let w, σ be two weights in $\mathbb{D}$ then

$$\sup_{z\in\mathbb{D}} B(w)(z)B(\sigma)(z) < \infty, \tag{1}$$

if and only if

$$P_B(\sigma\cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w). \tag{2}$$

A counterexample

Proposition

Let $w = (1-|z|^2)^2$ and σ be a weight, then the weights w and σ satisfies the Berezin condition if and only if

$$\int_{Q_I} \sigma \, dA \lesssim \frac{1}{\log \frac{2}{|I|}} \quad \text{for all arcs } I \subset \mathbb{T}.$$

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On the other hand,

Proposition

If $w(z) = (1-|z|^2)^2$, $z \in \mathbb{D}$, and σ a weight then $P_B(\sigma \cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w)$ if and only if σdA is a Carleson measure for the Dirichlet space.

Stegenga's counterexample fits in this framework.

Two weight Conjecture for the Bergman Projection

Conjecture (Two weight Conjecture for the Bergman Projection)

Let w, σ be two weights in $\mathbb{D}$, then the following are equivalent:

1. $P_B(\sigma\cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$
2. $$\|P_B(\sigma 1_{Q_I})\|_{L^2(w,\mathbb{D})} \leq C_0 \|1_{Q_I}\|_{L^2(\sigma,\mathbb{D})},$$
   and
   $$\|P_B^*(w 1_{Q_I})\|_{L^2(\sigma,\mathbb{D})} \leq C_0 \|1_{Q_I}\|_{L^2(w,\mathbb{D})},$$
   for all intervals $I \in \mathbb{T}$ and with constant C_0 uniform on I.

A counterexample to Sarason Conjecture

Lemma

Let $f \in L^2_a$, and let g be a Lipschitz analytic function in $\mathbb{D}$ with $|g(z)| \geq c(1-|z|)$, for some constant $c > 0$ and all $z \in \mathbb{D}$.
(i) If $fg \in H^\infty$ and

$$\int_{Q_I} |f|^2 dA \lesssim \frac{1}{\log \frac{2}{|I|}} \quad \text{for all arcs } I \subset \mathbb{T},$$

then the Berezin condition holds.
(ii) If $T_f T_g^*$ is bounded then $|f|^2 dA$ is a Carleson measure for the Dirichlet space.

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then the Berezin condition holds.
(ii) If $T_f T_g^*$ is bounded then $|f|^2 dA$ is a Carleson measure for the Dirichlet space.

The counterexample is based on Stegenga's example, the key to finding such a g is in Dyn′kin's work.

The two weight problem for P_B: the Sarason case

Theorem (Aleman, Pott, R.)

Let $f, g \in L^2_a(\mathbb{D})$ and consider the weights $\sigma = |g|^2$ and $w = |f|^2$. Then the following are equivalent

1. $P_B(\sigma\cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$
2. $$\|P_B^+(\sigma 1_{Q_I})\|_{L^2(w,\mathbb{D})} \leq C_0 \|1_{Q_I}\|_{L^2(\sigma,\mathbb{D})},$$
   and
   $$\|P_B^+(w 1_{Q_I})\|_{L^2(\sigma,\mathbb{D})} \leq C_0 \|1_{Q_I}\|_{L^2(w,\mathbb{D})},$$
   for all intervals $I \in \mathbb{T}$ and with constant C_0 uniform on I.

Proof strategy

1. Prove a two weight estimate for P_B^+
   1. Find a dyadic model for P_B^+
   2. Use the two weight result for dyadic positive operators (Nazarov-Treil-Volber, Lacey-Sawyer-Uriarte-Tuero)
2. Prove the equivalence of boundedness of P_B and P_B^+

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1. $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$,
2. $P_B^+(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$

Open questions

1. Characterize the weights w and σ for which $P_B(\sigma\cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$.

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1. Characterize the weights w and σ for which $P_B(\sigma\cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$.
2. Are there other applications to the two weight problem for the Bergman projection other than Sarason conjecture?

The end

MUCHAS GRACIAS!
THANK YOU!