The two weight problem for the Bergman projection and Sarason Conjecture (Joint work with A. Aleman and S. Pott)

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- $L^p(\mathbb{D}) := \{ f : \mathbb{C} \mapsto \mathbb{C} : \int_{\mathbb{D}} |f|^p dA < \infty \}.$
- The Bergman space $L^p_a(\mathbb{D}) := \{ f \in L^p(\mathbb{D}) : f \text{ is analytic} \}.$

The Bergman projection

Let P_B be the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. The operator is known as the Bergman Projection and it can be written as

$$P_B(f)(z) = \int_{\mathbb{D}} \frac{f(\xi)}{(1-z\bar{\xi})^2} dA(\xi),$$

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The main question

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Given w and v two function weights (positive, locally integrable functions), find necessary and sufficient conditions for the boundedness of the Bergman projection P in the corresponding weighted spaces, i.e.,

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An equivalent formulation:

$$P(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D}),$$

where $\sigma = v^{-1}$.



Toeplitz Operators on $L^2_a(\mathbb{D})$

Definition

Let $f \in L^{\infty}(\mathbb{D})$, we define the Toeplitz operator with symbol f as

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Remark:

- ① $T_g^* = T_{\bar{g}}$ and $T_f(h) = fh$ when f is analytic.
- 2 Toeplitz operators with analytic symbols are known to be bounded if and only if the symbol is bounded.

Sarason Conjecture on the Bergman space

Let $f, g \in L^2_a(\mathbb{D})$

Conjecture (Sarason)

$$T_f T_g^* : L_a^2(\mathbb{D}) \mapsto L_a^2(\mathbb{D}) \iff \sup_{z \in \mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty$$

where

$$B(h)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{h(\xi)}{|1 - \bar{\xi}z|^4} dA(\xi)$$

is the so called Berezin transform.

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The corresponding conjecture for H^2 is similar, one simply replaces the Berezin transforms by Poisson integrals.



The Berezin condition

The Berezin condition $\sup_{z\in\mathbb{D}} B(|f|^2)(z)B(|g|^2)(z)<\infty$ is inspired by the Békollé-Bonami condition

$$\sup_{I\subset\mathbb{T}}\frac{w(Q_I)}{|Q_I|}\left(\frac{w^{1-p'}(Q_I)}{|Q_I|}\right)^{p-1}<\infty,$$

where Q_I is the Carleson box associated to I, $Q_I := \{ re^{i\theta} : 1 - |I| < r < 1, e^{i\theta} \in I \}.$

Motivation: Sarason Conjecture on the Bergman space The two weight problem for the Bergman projection The Sarason case Some open questions

What was known

 Necessity of the Berezin condition was proved by Stroethoff and Zheng.

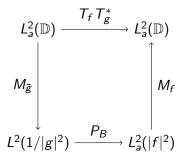
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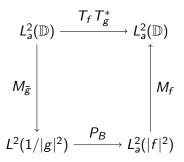
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- **1** In the H^2 case, the Poisson condition is necessary (Treil). But unfortunately not sufficient (Nazarov).

An observation by Cruz-Uribe



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Let
$$h \in L^2_a(\mathbb{D})$$
, $T_f T_g^*(h) = fP_B(\bar{g}h)$.



Generalized Sarason Conjecture

Conjecture (Two weight Conjecture for the Bergman Projection)

Let w, σ be two weights in \mathbb{D} then

$$\sup_{z\in\mathbb{D}}B(w)(z)B(\sigma)(z)<\infty,\tag{1}$$

if and only if

$$P_B(\sigma \cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w).$$
 (2)

A counterexample

Proposition

Let $w = (1 - |z|^2)^2$ and σ be a weight, then the weights w and σ satisfies the Berezin condition if and only if

$$\int_{Q_I} \sigma dA \lesssim \frac{1}{\log \frac{2}{|I|}} \quad \textit{for all arcs } I \subset \mathbb{T}.$$

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On the other hand,

Proposition

If $w(z) = (1 - |z|^2)^2$, $z \in \mathbb{D}$, and σ a weight then $P_B(\sigma \cdot) : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, w)$ if and only if σdA is a Carleson measure for the Dirichlet space.

Stegenga's counterexample fits in this framework.



Two weight Conjecture for the Bergman Projection

Conjecture (Two weight Conjecture for the Bergman Projection)

Let w, σ be two weights in \mathbb{D} , then the following are equivalent:

2

$$||P_B(\sigma 1_{Q_I})||_{L^2(w,\mathbb{D})} \le C_0 ||1_{Q_I}||_{L^2(\sigma,\mathbb{D})},$$

and

$$||P_B^*(w1_{Q_I})||_{L^2(\sigma,\mathbb{D})} \le C_0||1_{Q_I}||_{L^2(w,\mathbb{D})},$$

for all intervals $I \in \mathbb{T}$ and with constant C_0 uniform on I.

A counterexample to Sarason Conjecture

Lemma

Let $f \in L^2_a$, and let g be a Lipschitz analytic function in $\mathbb D$ with $|g(z)| \ge c(1-|z|)$, for some constant c>0 and all $z \in \mathbb D$. (i) If $fg \in H^\infty$ and

$$\int_{Q_I} |f|^2 dA \lesssim \frac{1}{\log \frac{2}{|I|}} \quad \text{for all arcs } I \subset \mathbb{T},$$

then the Berezin condition holds.

(ii) If $T_f T_g^*$ is bounded then $|f|^2 dA$ is a Carleson measure for the Dirichlet space.

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(ii) If $T_f T_g^*$ is bounded then $|f|^2 dA$ is a Carleson measure for the Dirichlet space.

The counterexample is based on Stegenga's example, the key to finding such a g is in Dyn'kin's work.



The two weight problem for P_B : the Sarason case

Theorem (Aleman, Pott, R.)

Let $f, g \in L^2_a(\mathbb{D})$ and consider the weights $\sigma = |g|^2$ and $w = |f|^2$. Then the following are equivalent

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$$||P_B^+(\sigma 1_{Q_I})||_{L^2(w,\mathbb{D})} \le C_0 ||1_{Q_I}||_{L^2(\sigma,\mathbb{D})},$$

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$$||P_B^+(w1_{Q_I})||_{L^2(\sigma,\mathbb{D})} \le C_0||1_{Q_I}||_{L^2(w,\mathbb{D})},$$

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Proof strategy

- Prove a two weight estimate for P_B^+
 - Find a dyadic model for P_B^+
 - Use the two weight result for dyadic positive operators (Nazarov-Treil-Volber, Lacey-Sawyer-Uriarte-Tuero)
- 2 Prove the equivalence of boundedness of P_B and P_B^+

Theorem (Aleman, Pott, R.)

Let $f, g \in L^2_a(\mathbb{D})$ and consider the weights $\sigma = |g|^2$ and $w = |f|^2$. Then the following are equivalent

- $\bullet P_B(\sigma \cdot): L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D}),$
- $P_{R}^{+}(\sigma \cdot): L^{2}(\sigma, \mathbb{D}) \mapsto L^{2}(w, \mathbb{D})$

Open questions

• Characterize the weights w and σ for which $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$.

Open questions

- **①** Characterize the weights w and σ for which $P_B(\sigma \cdot) : L^2(\sigma, \mathbb{D}) \mapsto L^2(w, \mathbb{D})$.
- ② Are there other applications to the two weight problem for the Bergman projection other than Sarason conjecture?

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The end

MUCHAS GRACIAS! THANK YOU!