Infinite-Dimensional Diagonalization

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Diagonal Matrices

A diagonal $n \times n$ matrix is of the following form:

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix},$$

where $a_{11}, \ldots, a_{nn}$ are coefficients in a field K.

- Geometrically, diagonal matrices correspond to *anisotropic scalings* (i.e., linear transformations that scale each axis direction but do not rotate it).
- The eigenvalues, inverses, powers, and determinants of a diagonal matrix are easy to compute.

Diagonalizable Matrices

Definition

A matrix $T \in \mathbb{M}_n(K)$ is *diagonalizable* if the following equivalent conditions hold.

(1) There is an invertible matrix $S \in \mathbb{M}_n(K)$ such that STS^{-1} is diagonal. (I.e., T is *similar* to a diagonal matrix.)

(2) There is a basis for K^n consisting of eigenvectors of T.

Fact

A matrix $T \in \mathbb{M}_n(K)$ is diagonalizable if and only if the minimal polynomial of T factors into linear terms over K and has no repeated roots.

Diagonalizable Algebras

Fact

The following are equivalent for any K-subalgebra $A \subseteq \mathbb{M}_n(K)$.

(1) A is simultaneously diagonalizable. (That is, there is an invertible matrix $S \in \mathbb{M}_n(K)$ such that SAS^{-1} consists of diagonal matrices.)

(2) $A \cong K^m$ as K-algebras, for some integer $1 \leq m \leq n$.

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Proof.

(1) $\Rightarrow$ (2) If $S \in \mathbb{M}_n(K)$ is such that SAS^{-1} consists of diagonal matrices, then conjugation by S gives an isomorphism $A \to K^{\dim A}$.

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(1) $\Rightarrow$ (2) If $S \in \mathbb{M}_n(K)$ is such that SAS^{-1} consists of diagonal matrices, then conjugation by S gives an isomorphism $A \to K^{\dim A}$.

(2) $\Rightarrow$ (1) If $A \cong K^m$, then A is spanned by an orthogonal set of idempotents $\{E_i \mid 1 \leq i \leq l\}$ with sum 1. It follows that $K^n = \bigoplus_{i=1}^{l} E_i(K^n)$ and each $E_i(K^n)$ is an eigenspace of each E_j, and hence of every matrix in A. □

Definitions and Notation

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- If $\dim_K(V) = n < \aleph_0$, then this is equivalent to there being an invertible matrix $S \in \mathbb{M}_n(K) = \mathrm{End}_K(V)$ such that SAS^{-1} consists of diagonal matrices.

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- If $\dim_K(V) = n < \aleph_0$, then this is equivalent to there being an invertible matrix $S \in \mathbb{M}_n(K) = \mathrm{End}_K(V)$ such that SAS^{-1} consists of diagonal matrices.
- Question: Can the above results about $\mathbb{M}_n(K)$ be directly generalized to $\mathrm{End}_K(V)$, for an arbitrary vector space V?

Example

Suppose that K is infinite and V is countably infinite-dimensional, with basis $\{v_i \mid i \in \mathbb{N}\}$. Letting the elements of $\mathrm{End}_K(V)$ act on V from the left, we may view them as column-finite matrices.

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Let $T \in \mathrm{End}_K(V)$ be the right shift, defined by $T(v_i) = v_{i+1}$, and let $S \in \mathrm{End}_K(V)$ be a diagonal matrix with distinct entries on the main diagonal. So

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } S = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Then $K[S]$ is diagonalizable, but S has no minimal polynomial. Also, $K[T]$ is not diagonalizable, since T has no eigenvectors. However, since neither T nor S satisfies any nonzero polynomial relations, $K[T] \cong K[x] \cong K[S]$ as K-algebras.

Function Topology

Definition

A basis of open sets for the *function topology* on $\mathrm{End}_K(V)$ is given by the sets

$$\{T \in \mathrm{End}_K(V) \mid T(x_1) = y_1, \ldots, T(x_l) = y_l\},$$

with $x_1, \ldots, x_l, y_1, \ldots, y_l \in V$.

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Facts

(1) $R = \mathrm{End}_K(V)$ is a topological ring with respect to the function topology. That is, $\cdot : R \times R \to R$, $+ : R \times R \to R$, and $- : R \to R$ are continuous.

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(2) The function topology on $\mathrm{End}_K(V)$ is Hausdorff and complete.

(3) If V is finite-dimensional, then the function topology on $\mathrm{End}_K(V) = \mathbb{M}_n(K)$ is just the discrete topology.

The Countable Case

Suppose that V is countably infinite-dimensional, with basis $\{v_i \mid i \in \mathbb{N}\}$.

- A typical basic open set in $\mathrm{End}_K(V)$ consists column-finite matrices of the form
$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$
where $A \in \mathbb{M}_n(K)$ is a fixed $n \times n$ matrix.

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$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$
where $A \in \mathbb{M}_n(K)$ is a fixed $n \times n$ matrix.
- The function topology on $\mathrm{End}_K(V)$ is induced by the following metric d. Given $T, S \in \mathrm{End}_K(V)$, let
$$d(T,S) = \begin{cases} 0 & \text{if } T = S \\ 2^{-(i+1)} & \text{if } T \neq S \end{cases},$$
where $i \in \mathbb{N}$ is the least number such that $T(v_i) \neq S(v_i)$.

Product Topology

Definition

For any cardinal Ω, a basis of open sets for the *product topology* on $K^\Omega = \prod_{i \in \Omega} K_i$ is given by the sets $\prod_{i \in \Omega} U_i$, where $U_i \subseteq K_i$ and $U_i \neq K_i$ for only finitely many $i \in \Omega$.

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(1) K^Ω is a topological ring with respect to the product topology.

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Proposition

If $A \subseteq K^\Omega$ is a closed K-subalgebra (in the product topology), then $A \cong K^\Gamma$, as topological K-algebras, for some cardinal $\Gamma \leq \Omega$.

Theorem

The following are equivalent for any closed K-subalgebra $A \subseteq \mathrm{End}_K(V)$.

(1) A is diagonalizable.

(2) $A \cong K^{\Omega}$ as topological K-algebras, for some cardinal $\Omega \leq \dim(V)$.

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Proof.

(2) $\Rightarrow$ (1) If $A \cong K^\Omega$, then there are orthogonal idempotents $E_i \in A$ $(i \in J)$ such that A is the closure of the K-subalgebra generated by $\{E_i \mid i \in J\}$, and for any finite-dimensional subspace $W \subseteq V$ there are $E_1, \dots, E_n \in \{E_i \mid i \in J\}$ such that $E_1 + \cdots + E_n$ agrees on W with the identity transformation. Thus $V = \bigoplus_{i \in J} E_i V$, and each $E_i V$ is an eigenspace of every element of A.

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(1) $\Rightarrow$ (2) Suppose that A is diagonalizable with respect to basis B of V. Letting $D = \{T \in \mathrm{End}_K(V) \mid T(v) \subseteq Kv \text{ for every } v \in B\}$, we have $A \subseteq D$. Define $\phi : D \to K^B$ by $\phi(T) = (\alpha_v)_{v \in B}$, where for each $v \in B$, $T(v) = \alpha_v v$. Then ϕ is a K-algebra isomorphism that is also a homeomorphism. Since A is closed, $\phi(A)$ must be a closed subalgebra of K^B. Such subalgebras are of the form K^Ω $(\Omega \leq \dim(V))$, by the previous Proposition. □

Diagonalizable Subalgebras

Theorem

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(1) A is diagonalizable.

(2) $A \cong K^{\Omega}$ as topological K-algebras, for some cardinal $\Omega \leq \dim(V)$.

Corollary

The following are equivalent for any K-subalgebra $A \subseteq \mathrm{End}_K(V)$.

(1) A is diagonalizable.

(2) $\overline{A}$ is diagonalizable.

(3) $\overline{A} \cong K^{\Omega}$ as topological algebras, for some cardinal $\Omega \leq \dim(V)$.

Diagonalizable Transformations

Theorem

Let $I \in \mathrm{End}_K(V)$ be the identity transformation. Then the following are equivalent for any $T \in \mathrm{End}_K(V)$.

(1) T is diagonalizable.

(2) For every open neighborhood $\mathcal{U}$ of 0 there are distinct $\lambda_1, \dots, \lambda_l \in K$ such that $(T - \lambda_1 I) \dots (T - \lambda_l I) \in \mathcal{U}$.

(3) For every finite-dimensional subspace $W \subseteq V$, there are distinct $\lambda_1, \dots, \lambda_l \in K$ such that $(T - \lambda_1 I) \dots (T - \lambda_l I)$ is zero on W.

(4) $\overline{K[T]} \cong K^{\Omega}$ as topological K-algebras, for some cardinal $\Omega \leq \dim_K(V)$.

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(2) $\Leftrightarrow$ (3) This follows from the definition of the function topology.

(1) $\Leftrightarrow$ (4) This follows from the previous corollary, since T is diagonalizable if and only if $K[T]$ is diagonalizable. □

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Proof.

(1) $\Rightarrow$ (2) Suppose that T is diagonalizable with respect to basis B of V, and let $\mathcal{U}$ be an open neighborhood of 0. Passing to a subset, we may assume that $\mathcal{U} = \{S \in \mathrm{End}(V) \mid S(u_1) = \cdots = S(u_n) = 0\}$ for some $u_1, \ldots, u_n \in V$. Then $\{u_1, \ldots, u_n\} \subseteq Kv_1 + \cdots + Kv_m$ for some $v_1, \ldots, v_m \in B$. Let $\lambda_i \in K$ be such that $T(v_i) = \lambda_i v_i$ $(1 \leq i \leq m)$. Upon reindexing, we may assume that $\lambda_1, \ldots, \lambda_l$ $(l \leq m)$ are the distinct elements of $\{\lambda_1, \ldots, \lambda_m\}$. Setting $S = (T - \lambda_1 I) \ldots (T - \lambda_l I)$, we have $S(Kv_1 + \cdots + Kv_m) = 0$, since the factors $T - \lambda_i I$ commute with each other, and hence $S \in \mathcal{U}$.

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(2) $\Rightarrow$ (1) Let $\Lambda = \{\lambda \in K \mid \ker(T - \lambda I) \neq 0\}$. Also, for each $\lambda \in \Lambda$ let B_λ be a basis for $\ker(T - \lambda I)$, and let $B = \bigcup_{\lambda \in \Lambda} B_\lambda$. Then B is a basis for V, with respect to which T is diagonalizable. □

Simultaneous Diagonalization

Fact

If $T, S \in \mathbb{M}_n(K)$ are both diagonalizable and commute, then they are simultaneously diagonalizable, i.e., $\{T, S\}$ is diagonalizable.

Since $\mathbb{M}_n(K)$ is finite-dimensional, it follows that any commuting set of diagonalizable matrices in $\mathbb{M}_n(K)$ is simultaneously diagonalizable.

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Theorem

Let $C, D \subseteq \mathrm{End}_K(V)$ be diagonalizable subalgebras that centralize one another (i.e., $TS = ST$ for all $T \in C$ and $S \in D$). Then the subalgebra $K[C \cup D]$ of $\mathrm{End}_K(V)$ generated by C and D is diagonalizable.

In particular, any finite set of commuting diagonalizable transformations in $\mathrm{End}_K(V)$ is simultaneously diagonalizable.

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which we denote by $E_1, E_2, E_3, \ldots$. Then $E_i^2 = E_i$, $E_i E_j = 0$ for $i \neq j$, and each E_i is diagonalizable. So the E_i span A as a K-vector space. Also $A = \overline{A}$.

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A commuting set of diagonalizable transformations in $\mathrm{End}_K(V)$ need not be simultaneously diagonalizable, when V is infinite-dimensional.

Suppose that V is countably infinite-dimensional, with basis $\{v_i \mid i \in \mathbb{N}\}$. Let $A \subseteq \mathrm{End}_K(V)$ be the subalgebra generated by the elements

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots,$$

which we denote by $E_1, E_2, E_3, \dots$. Then $E_i^2 = E_i$, $E_i E_j = 0$ for $i \neq j$, and each E_i is diagonalizable. So the E_i span A as a K-vector space. Also $A = \overline{A}$.

If the E_i were simultaneously diagonalizable, then A would be diagonalizable. Hence, by the main Theorem, we would have $\overline{A} \cong K^\Omega$ for some infinite cardinal Ω, since A is infinite-dimensional. But $\dim_K(\overline{A}) = \dim_K(A) = \aleph_0$, contradicting the fact that $\dim_K(K^\Omega)$ is uncountable.

The Closure

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Question: Is it the case that $\mathcal{D}_K(V) = \overline{\mathcal{D}_K(V)}$?

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Answer: That depends on the cardinality of K!

Proposition

Suppose that K is a finite field with q elements, and let $T \in \mathrm{End}_K(V)$. Then $T \in \mathcal{D}_K(V)$ if and only if $T^q = T$. Consequently, $\mathcal{D}_K(V) = \overline{\mathcal{D}_K(V)}$.

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Proof.

Every element of K is a zero of the polynomial $p(x) = x^q - x$. Thus, if T is diagonalizable with respect to basis B of V, then $T^q(v) = T(v)$ for all $v \in B$. It follows that $T^q = T$.

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Conversely, if $T^q = T$, then T has a minimal polynomial $\mu(x)$ that divides $p(x) = x^q - x = \prod_{\lambda \in K}(x - \lambda)$. Thus $\mu(x)$ factors into distinct linear terms over K, and hence T is diagonalizable, by a previous Theorem.

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Theorem

Suppose that K is infinite, and let $T \in \mathrm{End}_K(V)$. Also, let $H(T) \subseteq V$ be the set of all v such that $p(T)v = 0$ for some nonzero polynomial $p(x) \in K[x]$. Then $T \in \overline{\mathcal{D}_K(V)}$ if and only if T is diagonalizable on $H(T)$.

Example

Suppose that V is countably infinite-dimensional, with basis $\{v_i \mid i \in \mathbb{N}\}$. Let $T_R \in \mathrm{End}_K(V)$ be the right shift, defined by $T_R(v_i) = v_{i+1}$ for all i, and let $T_L \in \mathrm{End}_K(V)$ be the left shift, defined by $T_L(v_i) = v_{i-1}$ for $i \geq 1$, and $T_L(v_0) = 0$. Then

$$T_R = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } T_L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Neither T_R nor T_L is diagonalizable, since on the subspace of V spanned by $\{v_0, \ldots, v_n\}$, T_R satisfies no nonzero polynomial, while T_L satisfies $p(x) = x^n$. Thus, if K is finite, then $T_R, T_L \notin \mathcal{D}_K(V) = \overline{\mathcal{D}_K(V)}$.

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If K is infinite, then $T_L \notin \overline{\mathcal{D}_K(V)}$, since $H(T_L) = V$. But $T_R \in \overline{\mathcal{D}_K(V)}$, since $H(T_R) = 0$, and hence T_R is diagonalizable on $H(T_R)$.

Thank you!